

# Non-stationary de Sitter cosmological models

Ng. Ibohal

Department of Mathematics, Manipur University,

Imphal 795003, Manipur, INDIA.

E-mail: ngibohal@iucaa.ernet.in

## Abstract

In this note it is proposed a class of non-stationary de Sitter, rotating and non-rotating, solutions of Einstein's field equations with a cosmological term of variable function  $\Lambda^*(u)$ . It is found that the space-time of the rotating non-stationary de Sitter model is an algebraically special in the Petrov classification of gravitational field with a null vector, which is geodesic, shear free, expanding as well as non-zero twist. However, that of the non-rotating non-stationary model is conformally flat with non-empty space.

*Keywords:* Non-stationary de Sitter; rotating and non-rotating cosmological models; Kerr-Schild ansatz.

It is well known that the original de Sitter cosmological model is *conformally* flat  $C_{abcd} = 0$  space-time with *constant curvature*  $R_{abcd} = (\Lambda^*/3)(g_{ac}g_{bd} - g_{ad}g_{bc})$  [1]. It also describes the *non-rotating* and *stationary* solution. Therefore, the non-rotating stationary de Sitter model is a solution of Einstein's field equations for an empty space with constant curvature, whereas the *rotating* stationary de Sitter model proposed in Ref. 2 is a solution for *non-empty* space with *non-constant* curvature. Because of the stationary and non-rotating properties of the original de Sitter space, the non-rotating Schwarzschild black hole with constant mass can embed to produce Schwarzschild-de Sitter cosmological black hole with two event horizons - one for black hole and other for cosmological [3]. Similarly, the rotating stationary de Sitter cosmological universe [2] can conveniently embed into the rotating stationary Kerr-Newman solution to produce rotating Kerr-Newman-de Sitter cosmological black hole with constant cosmological term. This Kerr-Newman-de Sitter black hole metric can be expressed in terms of Kerr-Schild ansatz with different backgrounds as  $g_{ab}^{\text{KNdS}} = g_{ab}^{\text{dS}} + 2Q(r, \theta)\ell_a\ell_b$  where  $Q(r, \theta) = -(rm - e^2/2)R^{-2}$ , and  $g_{ab}^{\text{KNdS}} = g_{ab}^{\text{KN}} + 2H(r, \theta)\ell_a\ell_b$  with  $H(r, \theta) = -(\Lambda^*r^4/6)R^{-2}$ . Here  $g_{ab}^{\text{dS}}$  is the rotating stationary de Sitter metric and the vector  $\ell_a$  is a geodesic, shear free, expanding as well as non-zero twist, and one of the repeated principal null vectors of  $g_{ab}^{\text{KN}}$ ,  $g_{ab}^{\text{dS}}$  and  $g_{ab}^{\text{KNdS}}$ , as these space-times are Petrov type D. The expressibility of an embedded black hole in different Kerr-Schild ansatz means that, it is always true to talk about either Kerr-Newman black hole embedded into the rotating de Sitter space as Kerr-Newman-de Sitter or the rotating de Sitter space into Kerr-Newman black hole as rotating de Sitter-Kerr-Newman black hole - geometrically both are the same. That is, physically one may not be able to predict which space starts first to embed into what space. One thing we found from the study of Hawking's radiation of Kerr-Newman-de Sitter black hole [4], is that, there is no effect on the cosmological constant  $\Lambda^*$  during the evaporation process of electrical radiation. The cosmological constant  $\Lambda^*$  always remains unaffected in Einstein's

field equations during Hawking's radiation process. That is, unless some external forces apply to remove the cosmological term  $\Lambda^*$  from the space-time geometry, it continues to exist along with the electrically radiating objects, rotating or non-rotating. This means that it might have started to embed from the very early stage of the embedded black hole, and should continue to embed forever. It is noted that the Kerr-Newman-de Sitter black hole proposed in Ref. 2 is found different from the one obtained by Carter [5] in the terms involving cosmological constant.

The black hole embedded into de Sitter space plays an important role in classical general relativity that the cosmological constant is found present in the inflationary scenario of the early universe in a stage where the universe is geometrically similar to the original de Sitter space [6]. Also embedded black holes can avoid the direct formation of negative mass naked singularities during Hawking's black hole evaporation process [4]. It is also known that the rotating Vaidya-Bonnor black hole with variable mass  $M(u)$  and charge  $e(u)$  is a non-stationary solution. When  $M(u)$  and  $e(u)$  become constants, the rotating Vaidya-Bonnor black hole will reduce to the stationary Kerr-Newman black hole. If one wishes to study the physical properties of the gravitational field of a *complete non-stationary* embedded black hole, e.g. rotating non-stationary Vaidya-Bonnor-de Sitter (not discussed in this note), one needs to derive a new *rotating non-stationary* de Sitter model with a cosmological term of variable function  $\Lambda^*(u)$ . That is, an observer travelling in a non-stationary space-time must also be able to find a non-stationary cosmological de Sitter space to embed, having a similar space-time structure with time dependent functions.

In this view, it is proposed a rotating *non-stationary* de Sitter solution of Einstein's field equations with a cosmological term of variable function  $\Lambda^*(u)$  in this note. Using Newman-Penrose formalism [7], a class of rotating metric with a mass function  $M(u, r)$  has been discussed in Ref. 2, where the mass function is being expressed in terms of Wang-Wu function  $q_n(u)$  [8] as

$$M(u, r) \equiv \sum_{n=-\infty}^{+\infty} q_n(u) r^n.$$

For obtaining a rotating *non-stationary* de Sitter solution, we choose the Wang-Wu function as

$$q_n(u) = \begin{cases} \Lambda^*(u)/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 3, \end{cases} \quad (1)$$

such that

$$M(u, r) = \frac{1}{6} r^3 \Lambda^*(u). \quad (2)$$

Then using this mass function in the rotating metric presented in equation (6.4) of Ref. 2, we obtain a rotating metric, describing a *non-stationary* de Sitter model with cosmological term  $\Lambda^*(u)$  in the null coordinates  $(u, r, \theta, \phi)$  as

$$ds^2 = \left\{ 1 - \frac{r^4 \Lambda^*(u)}{3R^2} \right\} du^2 + 2du dr$$

$$\begin{aligned}
& +2a \frac{r^4 \Lambda^*(u)}{3R^2} \sin^2 \theta du d\phi - 2a \sin^2 \theta dr d\phi \\
& - R^2 d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta \right\} R^{-2} \sin^2 \theta d\phi^2, \tag{3}
\end{aligned}$$

where  $R^2 = r^2 + a^2 \cos^2 \theta$  and  $\Delta^* = r^2 - r^4 \Lambda^*(u)/3 + a^2$ . Here  $\Lambda^*(u)$  denotes an arbitrary non-increasing function of the retarded time coordinate  $u$  and  $a$  being a constant rotational parameter. When one sets the function  $\Lambda^*(u)$  to be a constant, the line element (3) will reduce to the *rotating stationary* de Sitter space-time [2]. The complex null vectors for the above metric can be chosen as follows:

$$\begin{aligned}
\ell_a &= \delta_a^1 - a \sin^2 \theta \delta_a^4, \\
n_a &= \frac{\Delta^*}{2R^2} \delta_a^1 + \delta_a^2 - \frac{\Delta^*}{2R^2} a \sin^2 \theta \delta_a^4, \\
m_a &= -\frac{1}{\sqrt{2}R} \left\{ -ia \sin \theta \delta_a^1 + R^2 \delta_a^3 + i(r^2 + a^2) \sin \theta \delta_a^4 \right\}.
\end{aligned} \tag{4}$$

Here  $\ell_a$ ,  $n_a$  are real null vectors and  $m_a$  is complex with the normalization conditions  $\ell_a n^a = 1 = -m_a \bar{m}^a$ . By virtue of Einstein's field equations, we calculate the energy-momentum tensor describing matter field for the non-stationary space-time as

$$T_{ab} = \mu^* \ell_a \ell_b + 2\rho^* \ell_{(a} n_{b)} + 2p m_{(a} \bar{m}_{b)} + 2\omega \ell_{(a} \bar{m}_{b)} + 2\bar{\omega} \ell_{(a} m_{b)}, \tag{5}$$

where

$$\begin{aligned}
\mu^* &= -\frac{r^4}{6KR^2R^2} \left\{ 2r\Lambda^*(u)_{,u} + a^2 \sin^2 \theta \Lambda^*(u)_{,uu} \right\}, \quad \rho^* = \frac{r^4}{KR^2R^2} \Lambda^*(u), \\
p &= -\frac{r^2 \Lambda^*(u)}{KR^2R^2} \left\{ r^2 + 2a^2 \cos^2 \theta \right\}, \quad \omega = -\frac{ia r^3 \sin \theta}{6\sqrt{2}KR^2R^2} (R - 3\bar{R}) \Lambda^*(u)_{,u}, \tag{6}
\end{aligned}$$

with the universal constant  $K = 8\pi G/c^4$ . The quantity  $\mu^*$  interprets as a null density based on the derivative of  $\Lambda^*(u)$ ;  $\rho^*$  and  $p$  are the density and pressure of the non-stationary matter field, and  $\omega$  represents the rotational force density determined by the rotational parameter  $a$  coupling with the derivative of  $\Lambda^*(u)$ . That is, for non-rotating model  $a = 0$ , the rotational density  $\omega$  will vanish, describing a non-rotating fluid model for non-stationary de Sitter universe. From (6) we find the equation of state (the ratio of the pressure to energy density) for the non-stationary rotating solution as  $w = p/\rho = -(r^2 + 2a^2 \cos^2 \theta)r^{-2}$  for non-zero rotational parameter  $a$ . This equation of state will take the value  $w = -1$  at the poles  $\theta = \pi/2$  and  $\theta = 3\pi/2$ , showing that the de Sitter solution (3) with non-constant  $\Lambda^*(u)$  describes a rotating non-stationary dark energy model possessing negative pressure.

The trace of energy momentum tensor  $T_{ab}$  (5) is found as

$$T = 2(\rho^* - p). \tag{7}$$

Here it is observed that  $\rho^* - p > 0$  for rotating non-stationary de Sitter model. The energy-momentum tensor (5) satisfies the energy conservation equations  $T^{ab}{}_{;b} = 0$ . The

verification of these equation may be seen in Appendix A below. The Ricci scalar  $\Lambda$  ( $\equiv \frac{1}{24}g^{ab}R_{ab}$ ), describing matter field by virtue of Einstein's field equations, is found as

$$\Lambda = \frac{1}{6}r^2\Lambda^*(u)R^{-2}. \quad (8)$$

Other Ricci scalars are related with  $\mu^*$ ,  $\rho^*$ ,  $p$  and  $\omega$  (6) as  $K\mu^* = 2\phi_{22}$ ,  $K\omega = -2\phi_{12}$ ,  $K\rho^* = 2\phi_{11} + 6\Lambda$  and  $Kp = 2\phi_{11} - 6\Lambda$ . The energy momentum tensor (5) also satisfies all the three energy conditions: (i) weak,  $T_{ab}U^aU^b \geq 0$ , (ii) strong,  $T_{ab}U^aU^b \geq \frac{1}{2}T$  and (iii) dominant, for a time-like observer with its four-velocity vector  $U^a$  as shown in Ref. 2. It is noted that  $T_{ab}$  (5) does not describe a perfect fluid, i.e. for a non-rotating perfect fluid,  $T_{ab}^{(\text{Pf})} = (\rho^* + p)u_a u_b - p g_{ab}$ , with unit time-like vector  $u_a$  and trace  $T^{(\text{Pf})} = \rho^* - 3p$ , which is different from the one given in (7).

The existing Weyl scalars, determining gravitational field for the space-time metric (3) are obtained as

$$\begin{aligned} \psi_2 &= \frac{r^2\Lambda^*(u)}{6\bar{R}\bar{R}R^2}\{(r + 2ia\cos\theta)\bar{R} - rR\}, \\ \psi_3 &= -\frac{ia r^3\sin\theta}{3\sqrt{2}\bar{R}\bar{R}R^2}(r + \bar{R})\Lambda^*(u)_{,u}, \\ \psi_4 &= \frac{a^2 r^4\sin^2\theta}{12\bar{R}\bar{R}R^2R^2}\{R^2\Lambda^*(u)_{,uu} - 2r\Lambda^*(u)_{,u}\}. \end{aligned} \quad (9)$$

From the non-vanishing Weyl scalars  $\psi_2$ ,  $\psi_3$  and  $\psi_4$ , it is observed that the rotating non-stationary de Sitter model (3) is an algebraically special (precisely, type II:  $C_{abc[d}\ell_{h]}\ell^b\ell^c = 0$ , with  $\psi_0 = \psi_1 = 0$  [9]) in the Petrov classification of space-time with a null vector  $\ell_a$  (4), which is geodesic, shear free, expanding ( $\hat{\theta} \equiv \frac{1}{2}\ell^a_{;a} = rR^{-2}$ ) as well as non-zero twist ( $\hat{\omega}^2 \equiv \frac{1}{2}\ell_{[a;b]}\ell^{a;b} = a^2\cos^2\theta R^{-2}R^{-2}$ ). Here the function  $\Lambda^*(u)$  does not involve in the expression of expansion  $\hat{\theta}$  and twist  $\hat{\omega}^2$  of the null vector  $\ell^a$ . This means that the physical properties of this null vector  $\ell^a$  are same for both *stationary* [2] as well as *non-stationary* (3) rotating de Sitter models, though they have different gravitational fields with different energy momentum tensors.

The expressions of  $\psi_3$  and  $\psi_4$  above involve the derivative of  $\Lambda^*(u)$  as  $\Lambda^*(u)_{,u}$ , coupling with the rotational parameter  $a$ . So at some point when  $\Lambda^*(u)$  sets to be a constant for non-zero rotation ( $a \neq 0$ ), both  $\psi_3$  and  $\psi_4$  will vanish. At that point the gravitational field of the observer will be of type  $D$  ( $\psi_2 \neq 0$ ) in the Petrov classification of stationary space-time. That is, the space-time becomes the rotating stationary de Sitter solution with cosmological constant  $\Lambda^*$  [2].

The metric (3) can be expressed in the coordinate system  $(t, x, y, z)$  as

$$\begin{aligned} ds^2 &= dt^2 - dx^2 - dy^2 - dz^2 \\ &+ \frac{\{r^6\Lambda^*(t, r)\}}{3(r^4 + a^2 z^2)} \left[ dt - \frac{1}{(r^2 + a^2)} \{r(xdx + ydy) + a(xdy - ydx)\} - \frac{1}{r} zdz \right]^2, \end{aligned}$$

where  $r$  is defined, in terms of  $x$ ,  $y$  and  $z$  [9]

$$r^4 - (x^2 + y^2 + z^2 - a^2)r^2 - a^2 z^2 = 0,$$

with the following transformations

$$\begin{aligned}x &= (r \cos\phi + a \sin\phi) \sin\theta, \\y &= (r \sin\phi - a \cos\phi) \sin\theta, \\z &= r \cos\theta, \quad t = u + r.\end{aligned}$$

Then, the above transformed metric is in the Kerr-Schild form with

$$g_{ab}^{\text{dS}} = \eta_{ab} + 2H(t, x, y, z) \ell_a \ell_b, \quad (10)$$

where  $\eta_{ab}$  is the flat metric and

$$\begin{aligned}H(t, x, y, z) &= -\frac{r^6 \Lambda^*(t, r)}{3(r^4 + a^2 z^2)}, \\ \ell_a dx^a &= dt - \frac{1}{(r^2 + a^2)} \{r(xdx + ydy) + a(xdy - ydx)\} - \frac{1}{r} z dz.\end{aligned}$$

In  $u$ -coordinate system the null vector  $\ell_a$  is given in (4) above. The Kerr-Schild ansatz (10) confirms that the rotating non-stationary de Sitter model (3) is a solution of Einstein's field equations of non-constant curvature.

The rotating non-stationary de Sitter solution has an apparent singularity when  $\Delta^* = r^2 - r^4 \Lambda^*(u)/3 + a^2 = 0$ . This equation has four roots  $r_{++}$ ,  $r_{+-}$ ,  $r_{-+}$  and  $r_{--}$ . They are found as

$$r_{\pm(\pm)} = \pm \sqrt{\frac{1}{2\Lambda^*(u)} \left\{ 3 \pm \sqrt{9 + 12a^2 \Lambda^*(u)} \right\}}. \quad (11)$$

Now let us denote these roots  $r_{++}$ ,  $r_{+-}$ ,  $r_{-+}$ ,  $r_{--}$  as  $r_1$ ,  $r_2$ ,  $r_3$ ,  $r_4$  respectively (for the simplicity,  $r_j$ ,  $j = 1, 2, 3, 4$ ). Then, these roots have the following relation

$$(r - r_1)(r - r_2)(r - r_3)(r - r_4) = -\frac{3}{\Lambda^*(u)} \left\{ r^2 - \frac{r^4 \Lambda^*(u)}{3} + a^2 \right\}.$$

Then each root represents the location of the cosmological horizon for the observer and associates an area of the horizon at each point at  $r = r_j$ ,  $j = 1, 2, 3, 4$ ,

$$\begin{aligned}\mathcal{A}_j &= \int_0^\pi \int_0^{2\pi} \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi \Big|_{r=r_j} \\ &= 4\pi \{r_j^2 + a^2\}.\end{aligned} \quad (12)$$

According to Bekenstein-Hawking area-entropy formula [3], these areas  $\mathcal{A}_j$  will determine the entropies  $\mathcal{S}_j$  of the horizons of the de Sitter model (3) by the relation  $\mathcal{S}_j = \mathcal{A}_j/4$  [3]. Thus, we find them as

$$\mathcal{S}_j = \pi \{r_j^2 + a^2\}. \quad (13)$$

The gravity of the cosmological horizons is determined by the surface gravity, defined by  $\kappa n^a = n^b \nabla_b n^a$  in [5], where the null vector  $n^a$  given in (4) above is parameterized by the coordinate  $u$ , such that  $d/du = n^a \nabla_a$ , and has the normalization condition  $\ell_a n^a = 1$

with the null vector  $\ell_a$ . The surface gravities  $\kappa_j$  associated at each  $r = r_j$  are found as below:

$$\kappa_p = \frac{r_p}{3R_p^2} \Lambda^*(u) \{r_p + r_{p+1}\} \{r_{p+1} - r_p\} \text{ for } p = 1, 3, \quad (14)$$

$$\kappa_p = \frac{r_p}{3R_p^2} \Lambda^*(u) \{r_{p-1} + r_p\} \{r_{p-1} - r_p\} \text{ for } p = 2, 4, \quad (15)$$

where  $R_j^2 = r_j^2 + a^2 \cos^2 \theta$ ,  $j = 1, 2, 3, 4$ . The surface gravities  $\kappa_j$  may be regarded as the gravitational field on the cosmological horizons. From (14) and (15), we observe that  $\kappa_1$  and  $\kappa_2$  will be zero when  $r_1$  and  $r_2$  coincide. Similarly,  $\kappa_3$  and  $\kappa_4$  will vanish when  $r_3 = r_4$ .

The coincidence of two roots  $r_1$  and  $r_2$  leads to a condition that  $\{9 + 12a^2 \Lambda^*(u)\} = 0$ . This condition implies that  $r_3$  and  $r_4$  also coincide. Then all roots take the form  $r_1 = r_2 = -r_3 = -r_4 = \sqrt{\{3/(2\Lambda^*(u))\}}$ . Accordingly, the area of the horizon at each point  $r_1, r_2, r_3$  and  $r_4$  are found as

$$\mathcal{A}_j = \frac{3\pi}{\Lambda^*(u)}. \quad (16)$$

This implies that the entropy associated with each point becomes

$$\mathcal{S}_j = \frac{3\pi}{4\Lambda^*(u)}, \quad j = 1, 2, 3, 4. \quad (17)$$

From this expression we find that the entropies  $\mathcal{S}_j$  at  $r_j$  are inversely proportional to the cosmological function  $\Lambda^*(u)$ . It is to mention that the value of  $\Lambda^*(u)$  is supposed to reduce according to the retarded time  $u$  change. Consequently the entropies  $\mathcal{S}_j$  may increase, as the function  $\Lambda^*(u)$  reduces. It is found that once the function  $\Lambda^*(u)$  takes the constant value, as in the case of *stationary* rotating de Sitter universe [2], the entropies  $\mathcal{S}_j$  associated with  $r_j$  will take constant values.

The angular velocities  $\Omega_j$  for the horizons are found as

$$\Omega_j = \lim_{r \rightarrow r_j} \left( -\frac{g_{u\phi}}{g_{\phi\phi}} \right) = -\frac{a r_j^4 \Lambda^*(u)}{3(r_j^2 + a^2)^2}. \quad (18)$$

The coincidences of the roots ( $r_1 = r_2$ , and  $r_3 = r_4$ ) imply that  $r_j = 3/(2\Lambda^*(u))$ . Then the angular velocities take the forms

$$\Omega_j = -\frac{4a}{3} \Lambda^*(u). \quad (19)$$

This indicates that angular velocities are directly proportional the cosmological function  $\Lambda^*(u)$ , and the effect of the change in  $\Lambda^*(u)$  will certainly affect on the angular velocities associated with  $r_j$ . It is also emphasized that the angular velocity  $\Omega_j$  given in (18) will vanish when the rotational parameter  $a$  tends to zero, showing the fact that there is no angular velocity for non-rotating de-Sitter space-time with vanishing rotational parameter  $a = 0$ .

It is quite interesting to discuss the nature of the non-rotating ( $a = 0$ ) non-stationary de Sitter model with  $\Lambda^*(u)$ . Although the non-rotating de Sitter model does not explain

the complete structure of the space-time, the metric is very simple without much mathematical expressions. Thus, when one sets the rotational parameter  $a$  to zero, the metric (3) reduces to non-rotating de Sitter model as

$$ds^2 = \left\{1 - \frac{1}{3}r^2\Lambda^*(u)\right\} du^2 + 2du dr - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (20)$$

In this situation, the Weyl scalars  $\psi_2$ ,  $\psi_3$  and  $\psi_4$  given in (9) are vanished showing that the space-time becomes the conformally flat ( $C_{abcd} = 0$ ). Then the energy-momentum tensor (5) takes the form

$$KT_{ab} = -\frac{1}{3}r\Lambda^*(u)_{,u}\ell_a\ell_b + \Lambda^*(u)g_{ab} \quad (21)$$

with its trace  $KT = 4\Lambda^*(u)$ . Here the energy-momentum tensor (21) involves a Vaidya-like null radiation term  $-\frac{1}{3}r\Lambda^*(u)_{,u}\ell_a\ell_b$ , which will vanish when  $r \rightarrow 0$ , and satisfies the energy conservation equation  $T^{ab}{}_{;b} = 0$ . However, it still maintains the non-stationary behavior  $\Lambda^*(u) \neq \text{constant}$ , showing that the space-time of the observer is naturally time dependent even at  $r \rightarrow 0$ . The energy momentum tensor (21) will become the one of the original de Sitter model [1] when  $\Lambda^*(u)$  takes a constant value. Using the energy momentum tensor (21) we find Einstein's field equations  $G_{ab} = -KT_{ab}$  for the non-rotating metric (20) as follows

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda(u)g_{ab} = -T_{ab}^{(\text{NS})}, \quad (22)$$

where the non-stationary evolution part of the energy-momentum tensor (20) is given by

$$T_{ab}^{(\text{NS})} = -\frac{1}{3}r\Lambda(u)_{,u}\ell_a\ell_b,$$

which has zero-trace  $T^{(\text{NS})} = 0$ . It is to mention that the universal constant  $K$  does not involve in the field equations (22). For future use we also present the null energy density  $\mu^*$ , energy density  $\rho^*$  and pressure  $p$  of the non-rotating de Sitter metric as follows

$$\mu^* = -\frac{r}{K}\Lambda^*(u)_{,u} \quad \rho^* = \frac{\Lambda^*(u)}{K}, \quad p = -\frac{\Lambda^*(u)}{K}. \quad (23)$$

From (23) we find the equation of state  $w = p/\rho^* = -1$  with the negative pressure of variable  $\Lambda^*(u)$ . It is also observed that the variable  $\Lambda^*(u)$  is the source of the energy density  $\rho^*$  and the negative pressure  $p$ , which determine the physical nature of the equation of state. For instance when we set it to zero, the density  $\rho^*$  and pressure  $p$  will be vanished, and consequently we cannot determine the dark energy equation of state parameter  $w = p/\rho^* = -1$ . This shows the fact that our non-stationary de Sitter solution (20) is in agreement with the cosmological constant ( $\Lambda^*$ ) de Sitter model possessing the equation of state  $w = -1$  in the dark energy scenario [10, 11, 12], when  $\Lambda^*(u)$  takes a constant value  $\Lambda^*$ .

The metric (20) has an apparent singularity with a horizon at  $r_{\pm} = \pm\{3\Lambda^{*(-1)}(u)\}^{1/2}$ . The entropies  $\mathcal{S}_{\pm}$  and the surface gravities  $\kappa_{\pm}$  of the horizon associated with  $r = r_{\pm}$  for

the non-rotating metric (20) are found as

$$\mathcal{S}_{\pm} = \frac{3\pi}{\Lambda^*(u)} \quad \text{and} \quad \kappa_{\pm} = \pm \sqrt{\frac{\Lambda^*(u)}{3}}. \quad (24)$$

From these expressions we observe that  $\mathcal{S}_{\pm}$  are inversely proportional to  $\Lambda^*(u)$ , whereas  $\kappa_{\pm}$  are not. It is emphasized that, according to the change of the retarded time  $u$ , the value of  $\Lambda^*(u)$  may decrease. This ensures that the entropies  $\mathcal{S}_{\pm}$  of the non-stationary de Sitter model will increase, whereas the surface gravities  $\kappa_{\pm}$  decrease. Such observations of changes of the values of  $\mathcal{S}_{\pm}$  and  $\kappa_{\pm}$  can be found only in the case of non-stationary de Sitter model (20). This is the important aim of the study of non-stationary de Sitter space-time. Our results of the non-rotating de Sitter solution are in agreement with those of Gibbons and Hawking [3], when the cosmological function  $\Lambda^*(u)$  tends to a constant  $\Lambda^*$ .

The Kretschmann scalar for non-rotating de Sitter model (17) takes the form

$$\mathcal{K} \equiv R_{abcd}R^{abcd} = \frac{8}{3}\Lambda^*(u)^2, \quad (25)$$

which does not involve any derivative term of  $\Lambda^*(u)$ , and will not change its value at  $r \rightarrow 0$  and  $r \rightarrow \infty$ . The above Kretschmann scalar will become the one of original de Sitter model when  $\Lambda^*(u)$  takes a constant value. That is, though the energy momentum tensor (21) for non-rotating non-stationary model is found different from the one of non-rotating stationary de Sitter solution, the forms of Kretschmann scalar for both *non-rotating non-stationary* (20) and *stationary* models [1] have similar structures with a difference in nature of the cosmological function  $\Lambda^*(u)$ .

Our result discussed here includes the following cosmological models: (i) original de Sitter when  $a = 0$ ,  $\Lambda^*(u) = \text{constant}$  [1], (ii) rotating stationary de Sitter when  $a \neq 0$ ,  $\Lambda^*(u) = \text{constant}$  [2], (iii) non-rotating non-stationary de Sitter when  $a = 0$ ,  $\Lambda^*(u) \neq \text{constant}$ , (iv) rotating non-stationary de Sitter when  $a \neq 0$ ,  $\Lambda^*(u) \neq \text{constant}$ . From the study of non-stationary models it is observed that the gravitational field of the space-time of the rotating model (3) is algebraically special in the Petrov classification whereas the non-rotating one (20) is conformally flat with the energy momentum tensor describing non-empty space. It is hoped that known stationary works, rotating or non-rotating, with cosmological constant, may be extended to the non-stationary ones by using the non-stationary de Sitter model, rotating (3) or non-rotating (20). It is to note that to the best of the author's knowledge, these non-stationary de Sitter cosmological solutions (3) and (20) have not been seen discussed before.

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## Appendix A: Energy-momentum tensor conservation equations

In this appendix we establish the fact that the energy-momentum tensor (5) satisfies the conservation equations  $T^{ab}{}_{;b} = 0$ . These are four equations, which can equivalently be expressed in three equations – two are real and one complex, using Newman-Penrose spin coefficients. Hence, we find the following

$$D\rho^* = (\rho^* + p)(\rho + \bar{\rho}) + \omega\bar{\kappa}^* + \bar{\omega}\kappa^*, \quad (\text{A1})$$

$$\begin{aligned} D\mu^* + \nabla\rho^* + \bar{\delta}\omega + \delta\bar{\omega} &= \mu^*\{(\rho + \bar{\rho}) - 2(\epsilon + \bar{\epsilon})\} - (\rho^* + p)(\mu + \bar{\mu}) \\ &\quad - \omega(2\pi + 2\bar{\beta} - \bar{\tau}) - \omega(2\bar{\pi} + 2\beta - \tau), \end{aligned} \quad (\text{A2})$$

$$D\omega + \delta p = \mu^*\kappa^* + (\rho^* + p)(\tau - \bar{\pi}) + \bar{\omega}\sigma - \omega(2\bar{\epsilon} - 2\rho - \bar{\rho}) \quad (\text{A3})$$

where  $\kappa^*$ ,  $\tau$ ,  $\pi$ , etc. are spin coefficients, and the derivative operators are defined by

$$D \equiv \ell^a \partial_a, \quad \nabla \equiv n^a \partial_a, \quad \delta \equiv m^a \partial_a, \quad \bar{\delta} \equiv \bar{m}^a \partial_a. \quad (\text{A4})$$

These are general equations for an energy-momentum tensor of the type (5).

Now, in order to verify whether components of  $T^{ab}$  with the quantities  $\mu^*$ ,  $\rho^*$ ,  $p$  and  $\omega$  given in (6) for the rotating non-stationary de Sitter metric satisfy these conservation equations (A1–A3) or not, we present the NP spin coefficients for the metric (3):

$$\begin{aligned} \kappa^* &= \sigma = \lambda = \epsilon = 0, \\ \rho &= -\frac{1}{\bar{R}}, \quad \mu = -\frac{\Delta^*}{2\bar{R}R^2}, \\ \alpha &= \frac{(2ai - R \cos \theta)}{2\sqrt{2}\bar{R}R \sin \theta}, \quad \beta = \frac{\cot \theta}{2\sqrt{2}R}, \\ \pi &= \frac{ia \sin \theta}{\sqrt{2}\bar{R}R}, \quad \tau = -\frac{ia \sin \theta}{\sqrt{2}R^2}, \\ \gamma &= \frac{1}{2\bar{R}R^2} \left[ \left\{ r - \frac{2}{3}r^2\Lambda^*(u) \right\} \bar{R} - \Delta^* \right], \\ \nu &= \frac{1}{6\sqrt{2}\bar{R}R^2} iar^4 \sin \theta \Lambda^*(u)_{,u} \end{aligned} \quad (\text{A5})$$

The derivative operators (A4) are given as follows:

$$\begin{aligned} D &= \partial_r, \\ \nabla &= \frac{1}{R^2} \left\{ (r^2 + a^2) \partial_u - \frac{\Delta^*}{2} \partial_r + a \partial_\phi \right\}, \\ \delta &= \frac{1}{\sqrt{2}R} \left\{ ia \sin \theta \partial_u + \partial_\theta + \frac{i}{\sin \theta} \partial_\phi \right\}, \end{aligned} \quad (\text{A6})$$

where  $\Delta^* = r^2 - r^4 \Lambda^*(u)/3 + a^2$ . The equations (A1) and (A3) are comparatively easier to verify than (A2). Therefore, we shall not show their verification here except for the equation (A2). Now, by virtue of (6) and (A6), the left side of (A2) is found as

$$\begin{aligned}
D\mu^* + \nabla\rho^* + \bar{\delta}\omega + \delta\bar{\omega} &= \frac{r^5 a^2 \sin^2 \theta}{3KR^2 R^2 R^2} \Lambda^*(u)_{,uu} - \frac{2r^3 a^2 \cos^2 \theta \Delta^*}{KR^2 R^2 R^2 R^2} \Lambda^*(u) \\
&+ \left\{ \frac{2r^6}{3KR^2 R^2 R^2} - \frac{a^2 r^4}{3KR^2 R^2 R^2} \right. \\
&\left. + \frac{2a^2 r^4 \sin^2 \theta}{3KR^2 R^2 R^2 R^2} (3a^2 \cos^2 \theta + r^2) \right\} \Lambda^*(u)_{,u}. \quad (\text{A7})
\end{aligned}$$

which can be shown equal to the right side of (A2), by using (6) and (5A). This leads to the conclusion of the verification that the energy-momentum tensor (5) satisfies the condition  $T^{ab}{}_{;b} = 0$ .

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