A Rough Guide to Stellar Structure

Hydrostatic equilibrium n non-relativistic stars are governed by the equations

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

with

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

In this rough guide, we look at only some dimensional, order-of-magnitude estimates. We approximate dP/dr as $(P_s - P_c)/R$, where P_c and P_s are the central and surface pressures of the star respectively, and R is the radius of the star. Writing the mass of the star as $M = 4\pi R^3 \rho/3$ and noting that normally $P_c \gg P_s$, we obtain

$$\frac{P_c}{R} = \frac{GM\rho}{R^2}$$

and hence

$$P_c = \frac{GM\rho}{R} = \left(\frac{4\pi}{3}\right)^{1/3} GM^{2/3}\rho^{4/3}$$

where we have eliminated R in the final expression.

White Dwarfs

In case of White Dwarfs, the pressure P_c is given by the electron degeneracy pressure, with the dependence on ρ as noted above. This pressure as a function of density can be represented in a graphical form, in a log-log plot, as in fig. 1(a). The low-density asymptote of this has a slope of 5/3 and the high-density asymptote a slope of 4/3.

In the same plot, we can also represent the requirement of the hydrostatic equilibrium, $P_c \propto M^{2/3}\rho^{4/3}$, as straight lines of slope 4/3, and intercept increasing with the mass of the star (see fig. 1(b)). The crossing of such a line for a given mass with the degeneracy pressure line represents an equilibrium solution for a white dwarf. Clearly, if the mass exceeds a certain limit, the curves would not cross, since the high density asymptote of the degeneracy pressure becomes parallel to the P_c lines. The maximum mass that can be supported is known as the *Chandrasekhar limit*, which, from rigorous numerical solution of the hydrostatic equilibrium equation, works out to be

$$M_{\rm ch} = \frac{5.836 M_\odot}{\mu_e^2}$$

A white dwarf is produced at the end of the nuclear burning of a normal star, and therefore is composed primarily of helium or heavier elements. For these elements, $\mu_e = 2$, and hence the limiting mass of a white dwarf is $1.46 M_\odot$.

At masses much less than the limiting mass, the equilibrium is obtained in the non-relativistic part of the equation of state, leading to

$$M^{2/3}\rho^{4/3} \propto \rho^{5/3}$$

and hence to

$$R \propto M^{-1/3}$$

i.e. the heavier the white dwarf, the smaller is its radius.

Conclusions can be drawn regarding other varieties of compact stars, e.g. Neutron Stars, in a similar manner (see fig. 1), but GR effects need to be also accounted for.

Main Sequence Stars

We have discussed above the hydrostatic equilibrium of compact stars. We now discuss normal stars, where gravity is balanced by thermal pressure. Using the dimensional techniques illustrated above, one concludes that for such stars

$$\frac{\rho k T}{\mu m_p} \approx G M^{2/3} \rho^{4/3}$$

or

$$T = \frac{\mu m_p G}{k} M^{2/3} \rho^{1/3}$$

The problem with thermal pressure supported configurations is that as the heat leaks out by radiation into the surroundings, the configuration must collapse unless the lost heat is replenished by energy generation in the interior. In the normal stars we see in the sky this energy generation takes place due to nuclear fusion. Since the nuclear energy source is not eternal, the equilibrium in a normal star is only a temporary respite in the continuing process of its gravitational collapse.

A star is born through the gravitational contraction of a protostellar cloud. As the cloud condenses, gravitational binding energy is released and radiated away. When the collapsing cloud is optically thick, the collapse proceeds slowly, through a sequence of states in virial equilibrium with zero surface pressure, i.e. the internal thermal energy accounting for half the gravitational binding energy:

$$E_{\rm th} = -\frac{1}{2}E_{\rm pot}$$

or

$$\frac{MkT}{\mu m_p} \approx \frac{GM^2}{R}$$

giving

$$RT = \frac{\mu m_p G}{k}M$$

as obtained also via the pressure balance condition above. This shows that as the contraction proceeds, the temperature of the gas cloud rises, and eventually it becomes hot enough to start burning hydrogen at the core. The temperature and density at the core adjusts itself to balance the loss of energy due to radiation from the surface by the energy generation due to nuclear fusion.

All such hydrogen burning stars have nearly the same core temperature, within about a factor of two, since the nuclear fusion rate is a very strong function of temperature. The central temperature at Hydrogen burning is $T_c = T_{\rm H} \approx 10^7$ K. It then follows that the radius of hydrogen burning stars are roughly proportional to their mass, as $R \propto M/T$. Since the fusion of hydrogen generates the largest amount of energy per unit mass of all possible burning stages, the star spends the largest fraction of its life in a hydrogen-burning stage. The collection of all such Hydrogen-burning stars is called the “Main Sequence”.

The luminosity of a star is decided by the rate of escape of radiation from the interior. Roughly speaking, the luminosity can be written as

$$L = \frac{\text{Radiative energy content}}{\text{Photon leakage time}}$$

We note that the total radiative energy content is

$$U_{\rm rad} \approx \frac{4\pi}{3} R^3 (aT^4)$$

The photon leakage time can be obtained from the following argument. A photon random walks in the optically thick interior before leaking out. Let the mean free path for this be l. The net displacement after N free paths is $l\sqrt{N}$. For this to equal the stellar radius R, $N = (R/l)^2$; and since passage through a free path requires a time l/c, this implies a photon leakage time

$$t_{\rm ph} \approx \frac{l}{c}\left(\frac{R}{l}\right)^2 = \frac{R^2}{lc}$$

Hence the luminosity works out to be

$$L = \frac{U_{\rm rad}}{t_{\rm ph}} = \frac{4\pi ac}{3} lRT^4$$

On the main sequence, $T \propto M/R$, and hence

$$L \propto l\frac{M^4}{R^3}$$

Now, the mean free path

$$l = \frac{1}{\alpha} = \frac{1}{n\sigma}$$

where α is the effective absorption coefficient, n is the number density of particles and σ the absorption/scattering cross-section of each particle.

At relatively high temperatures, i.e. for higher mass stars, electron scattering is the only dominant opacity source, for which σ is the Thompson cross section, independent of temperature. Hence $l \propto 1/\rho$ and $L \propto M^4/(\rho R^3) \propto M^3$. At somewhat lower temperatures, l obtained from the Rosseland Mean opacity goes as $T^{3.5}/\rho^2$, leading to a luminosity scaling of $L \propto M^{5.5}R^{-0.5}$. On an average, then, on the Main Sequence, $L \propto M^4$, and since the total available hydrogen fuel is proportional to M, the Main Sequence Lifetime of a star is $t_{\rm MS} \propto M^{-3}$.

The effective surface temperature of a star can be obtained from its Luminosity and radius, using the black body relation:

$$L = 4\pi R^2 \sigma T_{\rm eff}^4$$

where σ is the Stefan's constant. Since both L and R, on the main sequence, are monotonic functions of mass M, the main sequence stars lie in a thin band in a $\log L$ vs $\log T_{\rm eff}$ plot, known as the Hertzsprung-Russell diagram, or HR diagram for short (Fig. 2). Observationlly, the effective temperature is obtained using spectral measurements, and in the horizontal axis of the HR diagram is plotted the "spectral class" of stars, which are designated O,B,A,F,G,K,M in decreasing order of temperature. The sun happens to be a G star on the main sequence, with $T_{\rm eff} \approx 5700$ K.

In the HR diagram in fig. 2, apart from the very prominent main sequence, other important classes of stars, namely giants, supergiants and white dwarfs are also indicated. The non-main sequence stars are in stages other than core hydrogen burning. We have discussed white dwarfs already, and know that they are end states of stellar evolution with no further gravitational collapse possible unless mass is added to them. The remaining classes mentioned above are thermal pressure supported, and are hence transitory phases. The relative number of stars in different phases signify the relative durations of these phases. However, some phases such as supergiant can be accessed only by relatively massive stars, which come in smaller numbers to begin with. In a galactic census, the estimated number of stars of various categories in our galaxy would be about the following.

Luminosity Class	Spectral Class	Typical Mass ($M_\odot$)	Typical Luminosity ($L_\odot$)	Number of stars
Supergiant	O–M	$> 5?$	$\sim 30,000$	$\sim 10^5$
Giant	F–M	~ 1.2	~ 100	$\sim 2 \times 10^9$
Main Sequence	O	~ 25	$\sim 30,000$	$\sim 10^4$
	B	~ 5	~ 200	$\sim 3 \times 10^8$
	A	~ 1.7	~ 6	$\sim 3 \times 10^9$
	F	~ 1.2	~ 1.4	$\sim 1.2 \times 10^{10}$
	G	~ 0.9	~ 0.6	$\sim 2.6 \times 10^{10}$
	K	~ 0.5	~ 0.2	$\sim 5.2 \times 10^{10}$
	M	~ 0.25	~ 0.005	$\sim 2.7 \times 10^{11}$
White Dwarf	B–F	~ 0.6	~ 0.005	$\sim 3.5 \times 10^{10}$

The range of masses for which stars are found to exist in the Galaxy is $\sim 0.1 M_\odot$ to $\sim 100 M_\odot$. The lower mass limit is set by the configuration that becomes degeneracy pressure supported before being able to ignite hydrogen. As can be seen from the scaling relations above, on the Main Sequence the central density of a star is proportional to M^{-2}, so the lower the mass of the star the larger is the ratio of degeneracy pressure to thermal pressure at the hydrogen burning temperature $T_{\rm H}$. Below $\sim 0.08 M_\odot$, the contracting gas cloud reaches pressure equilibrium with electron degeneracy pressure while the central temperature $T_c (\propto M/R)$ is still below $T_{\rm H}$. Configurations below this limit are called *Brown Dwarfs*, which, for a brief period in their life history, undergo deuterium burning before settling onto the degenerate configuration. At masses below $\sim 10^{-2} M_\odot$ electrostatic forces, such as van der Waal's forces, provide the major support mechanism and objects of this kind are the planets.

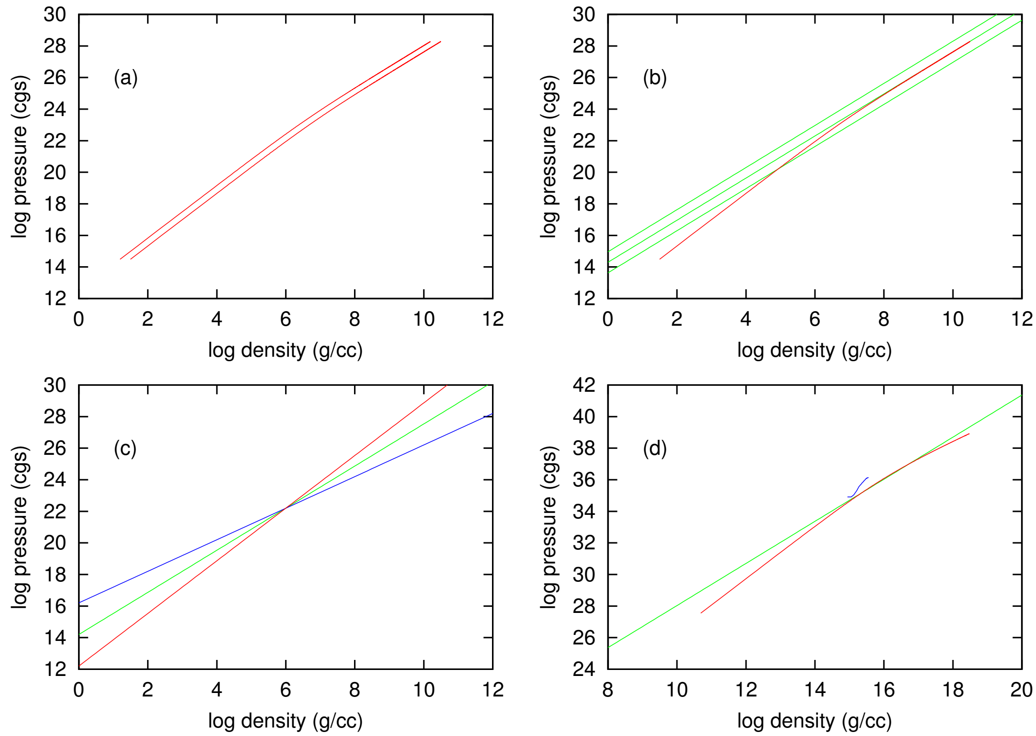

Figure 1: (a) electron degeneracy pressure as a function of density for $\mu_e = 1$ (upper curve) and 2 (lower curve). (b) P_c vs ρ required for hydrostatic equilibrium (green lines) plotted with the electron degeneracy pressure (red line). A higher mass corresponds to an upward shift of the green line. Equilibrium requirement for three different masses are shown. The middle line just grazes the upper part of the degeneracy pressure line, and represents the limiting mass of a white dwarf. (c) Stable and Unstable equilibria: a pressure curve (red line) steeper than the green line required by gravity leads to a stable equilibrium, and a flatter one (blue line) leads to an unstable equilibrium. (d) The red line here represents the degeneracy pressure of neutrons. Note that at high densities the pressure becomes proportional to ρ, as neutrons become relativistic. The crossing of a green line on this part of the pressure curve would represent an unstable equilibrium. The blue line represents a neutron star equation of state including the effects of repulsive strong interaction.

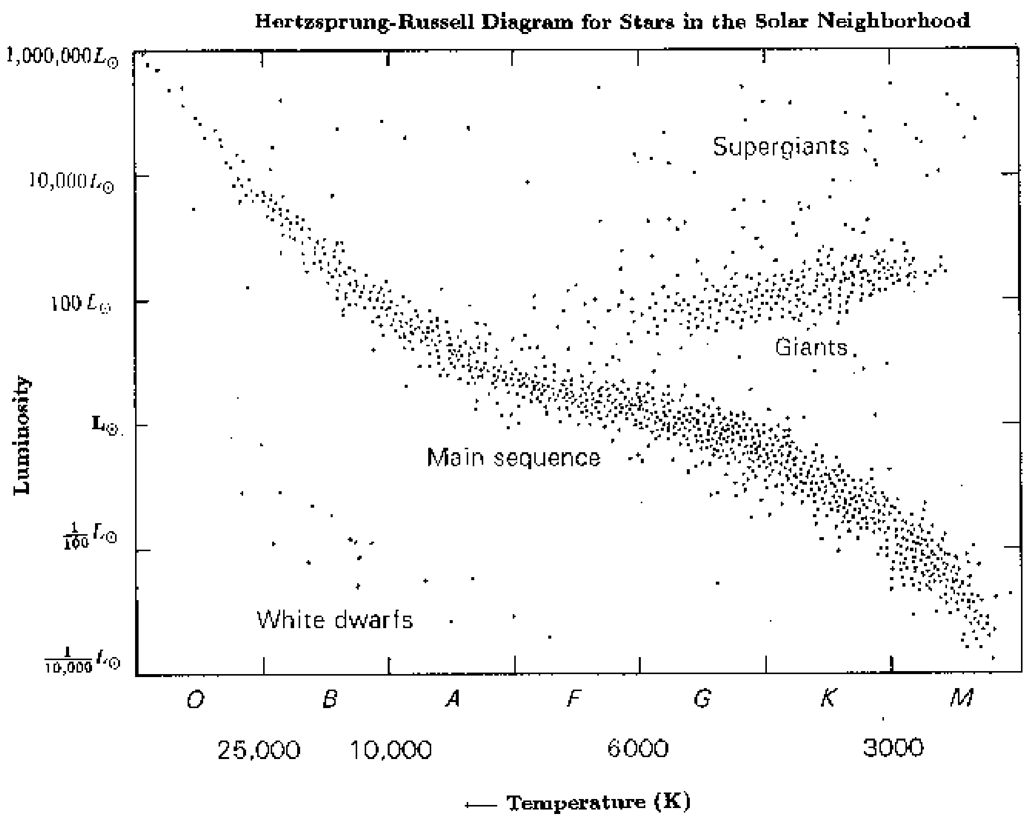

Figure 2: The Hertzsprung-Russell (HR) diagram of stars in the solar neighbourhood, with major luminosity classes indicated. For recent HR diagrams constructed from observations by the Hipparcos satellite, visit the website http://astro.estec.esa.nl/Hipparcos/TOUR/tour-hrdiagram.html