

Retarded Potentials and Radiation

Siddhartha Sinha
Joint Astronomy Programme student
Department of Physics
Indian Institute of Science
Bangalore.

December 10, 2003

Abstract

The transition from the study of electrostatics and magnetostatics to the study of accelerating charges and changing currents, leads to radiation of electromagnetic waves from the source distributions. These come as a direct consequence of particular solutions of the inhomogeneous wave equations satisfied by the scalar and vector potentials.

1 MAXWELL'S EQUATIONS

The set of four equations

$$\nabla \cdot \mathbf{D} = 4\pi\rho \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \quad (3)$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \quad (4)$$

known as the **Maxwell equations**, forms the basis of all classical electromagnetic phenomena. Maxwell's equations, along with the Lorentz force equation and Newton's laws of motion provide a complete description of the classical dynamics of interacting charged particles and electromagnetic fields. Gaussian units are employed in writing the Maxwell equations. These equations consist of a set of **coupled first order linear partial differential equations** relating the various components of the electric and magnetic fields. Note that Eqns (2) and (3) are **homogeneous** while Eqns (1) and (4) are **inhomogeneous**. The homogeneous equations can be used to define **scalar** and **vector potentials**. Thus Eqn(2) yields

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (5)$$

where \mathbf{A} is the vector potential. Substituting the value of \mathbf{B} from Eqn(5) in Eqn(3) we get

$$\nabla \times \left(\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = \mathbf{0} \quad (6)$$

It follows that

$$\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi \quad (7)$$

where ϕ is the scalar potential. The definition of \mathbf{B} and \mathbf{E} in terms of the potentials \mathbf{A} and ϕ satisfy identically the two homogeneous Maxwell Eqns. The dynamic behavior of \mathbf{A} and ϕ will be determined by the two inhomogeneous equations, i.e. Eqns (1) and (4). These two equations, written in terms of the potentials read

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t}) = -4\pi\rho \quad (8)$$

$$\nabla \times (\nabla \times \mathbf{A}) - \frac{1}{c} \frac{\partial}{\partial t} (-\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}) = \frac{4\pi}{c} \mathbf{J} \quad (9)$$

With the vector identity $\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$, Eqn (9) becomes

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t}) = -\frac{4\pi}{c} \mathbf{J} \quad (10)$$

Thus the set of four Maxwell Eqns have been reduced to a set of **two linear, second order ,coupled,partial differential equations**,by introducing the scalar and vector potentials. To **uncouple** them we use the **gauge invariance** of electromagnetic fields.Since \mathbf{B} is defined through Eqn(5) in terms of \mathbf{A} ,the vector potential is arbitrary to the extent that the gradient of some scalar function Λ can be added.Thus \mathbf{B} is left unchanged by the transformation, $\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\Lambda$ The electric field will also be unchanged if at the same time ϕ is changed by $\phi \rightarrow \phi' = \phi - \frac{1}{c} \frac{\partial \Lambda}{\partial t}$ The freedom implied by the above two gauge transformations means that we can choose a set of potentials(\mathbf{A}, ϕ)such that

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \quad (11)$$

This is called the **Lorentz condition**.Under this condition the pair of equations (8) and (10) are uncoupled ,leaving us with **two inhomogeneous wave equations**;one each for \mathbf{A} and ϕ .

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho \quad (12)$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{J} \quad (13)$$

2 SOLUTIONS OF THE WAVE EQUATIONS

As we know, the solution of the inhomogeneous linear equations (12)and (13) can be represented as the sum of the solutions of these equations without the right-hand side,and a particular integral of these equations with the right-hand side.To find the particular solution, we divide the whole space into infinitely small regions and determine the field produced by the charges in one of these volume elements.Because of the **linearity** of the wave equations ,the **Superposition Principle** is valid and the actual field will be the sum of the fields produced by all such elements. The charge de in a given volume element is,generally speaking, a function of time.If we choose the origin of coordinates in the volume element under consideration, then the charge density is $\rho(t) = de(t)\delta(\mathbf{R})$, where \mathbf{R} is the distance from the origin.Thus we must solve the equation

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi de(t)\delta(\mathbf{R}) \quad (14)$$

for the scalar potential ϕ . Everywhere, except at the origin, $\delta(\mathbf{R}) = 0$, and we have the equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \quad (15)$$

It is clear that in the case we are considering ϕ has spherical symmetry, i.e., ϕ is a function of \mathbf{R} only. Expressing the Laplacian operator in spherical polar coordinates, Eqn(15) reduces to

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\right) = 0$$

To solve this Equation, we make the substitution

$$\phi = \frac{\chi(R,t)}{R}.$$

Then we find for χ

$$\frac{\partial^2\chi}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0 \quad (16)$$

But this is the equation of travelling plane waves, whose solution has the form

$$\chi = f_1\left(t - \frac{R}{c}\right) + f_2\left(t + \frac{R}{c}\right)$$

Since we only want a particular solution, it is sufficient to choose only one of the functions f_1 and f_2 . Usually it is convenient to choose $f_2 = 0$ (reason given later). Then, everywhere except at the origin, ϕ has the form

$$\phi = \frac{\chi\left(t - \frac{R}{c}\right)}{R} \quad (17)$$

So far χ is arbitrary; we now choose it so that we get the correct value at the origin, i.e. equation(14) must be satisfied. Now, as $R \rightarrow 0$, the potential increases to infinity, and therefore its derivatives with respect to the coordinates increase more rapidly than its time derivative. Thus we can in Eqn (14), neglect the term $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$ compared with $\nabla^2\phi$. Then Eqn(14) goes over to the Poisson Equation, leading to the Coulomb law. Thus, near the origin, Eqn(14) must go over into the Coulomb law, from which it follows that $\chi(t) = de(t)$, that is,

$$\phi = \frac{de\left(t - \frac{R}{c}\right)}{R} \quad (18)$$

To find ϕ for an arbitrary distribution of charges $\rho(x, y, z, t)$, write in Eqn(18) $de = \rho dV$, where dV is the volume element and integrate over all space. To this solution of the inhomogeneous equation() we can still add the solution ϕ_0 of the same equation without the right-hand side. Thus the general solution has the form:

$$\phi(x, y, z, t) = \int \frac{1}{R}\rho(x', y', z', t - \frac{R}{c})dV' + \phi_0 \quad (19)$$

where $R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$, and $dV' = dx' dy' dz'$ Similarly for \mathbf{A} , we get

$$\mathbf{A}(x, y, z, t) = \int \frac{1}{R} \mathbf{J}(x', y', z', t - \frac{R}{c}) dV' + \mathbf{A}_0 \quad (20)$$

The potentials in Eqns (19) and (20), without \mathbf{A}_0 and ϕ_0 are called the **Retarded Potentials** (for reasons to be given later). The quantities \mathbf{A}_0 and ϕ_0 in (19) and (20) must be such that the conditions of the problem are fulfilled. To do this it is sufficient to impose initial conditions, i.e., to fix the values of the fields at the initial time. However we do not usually have to deal with such conditions. Instead we are usually given conditions at large distances from the system of charges throughout all of time. Thus, we may be told that radiation is incident on the system from outside. Corresponding to this, the field which is developed as a result of the interaction of this radiation with the system can differ from the external field only by the radiation originating from the system. This radiation must, at large distances, have the form of waves spreading out from the system, i.e., in the direction of increasing R . But precisely this condition is satisfied by the retarded potentials. Thus these solutions represent the field produced by the system, while \mathbf{A}_0 and ϕ_0 must be set equal to the external field acting on the system.

3 THE RETARDED AND ADVANCED POTENTIALS

More generally the retarded potentials can be written as

$$\phi(\mathbf{r}, t) = \int \frac{[\rho] d^3 \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \quad (21)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int \frac{[\mathbf{J}] d^3 \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \quad (22)$$

The notation $[Q]$ means that Q is to be evaluated at the **retarded time**

$$[Q] = Q(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}) \quad (23)$$

Here $|\mathbf{r} - \mathbf{r}'|$ is the distance from the source point \mathbf{r}' to the field point \mathbf{r} . Now electromagnetic "news" travel at the speed of light. When the source charge and current distributions are time dependant, it is not the instantaneous status of the source distribution that matters, but rather its condition at some earlier time t_{ret} (called the **retarded time**) when the "message" left. Since this message must travel a distance $|\mathbf{r} - \mathbf{r}'|$, the delay is $\frac{|\mathbf{r} - \mathbf{r}'|}{c}$. Because the

integrands in (21) and (22) are evaluated at the retarded time ,these are called **retarded potentials**.Note that the retarded time is not the same for all points of the source distribution,the most distant parts of the source have earlier retarded times than the nearby ones.Also note that the retarded potentials reduce properly to

$$\phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \quad (24)$$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \quad (25)$$

A similar set of solutions with the potentials evaluated at the **advanced**time $t_{adv} = t + \frac{|\mathbf{r} - \mathbf{r}'|}{c}$ are also mathematically valid solutions of the inhomogeneous wave equations() and ()(see $f_2(t + \frac{R}{c})$ in section 2).These are accordingly called the **advanced potentials**.Although the advanced potentials are entirely consistent with Maxwell's equations, they violate the most sacred tenet in all of physics:the **Principle of Causality**.they suggest that the potentials now depend on what the charge and current distribution will be at some time in the future-the effect, in other words, precedes the cause.Although the advanced potentials are of some theoretical interest,they have no direct physical signifance.(Because the D'Alembertian involves t^2 instead of t , the theory itself is **time reversal invariant**,and does not distinguish "past" from "future".Time assymetry is introduced when we select the retarded potentials in preference to the advanced ones, reflecting the reasonable belief that electromagnetic influences propagate forward,not backward, in time).

4 LIENARD-WIECHART POTENTIALS

We now calculate the potentials $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$,of a point charge q that is moving on a specified trajectory: $\mathbf{w}(t) \equiv$ position of q at time t . The retarded time is determined implicitly by the equation

$$|\mathbf{r} - \mathbf{w}(t)| = c(t - t_{ret}) \quad (26)$$

for the left side is the distance the "news" travel, and $(t - t_{ret})$ is the time it takes to make the trip(see figure). $\mathbf{w}(t_{ret})$ is called the **retarded position** of the charge; \mathbf{R} is the vector from the retarded position to the field point \mathbf{r} :

$$\mathbf{R} = \mathbf{r} - \mathbf{w}(t_{ret}) \quad (27)$$

It is important to note that *at most one point is "in communication"* with \mathbf{r} at *any particular time* t . For suppose there were two such points, with retarded

times t_1 and t_2 : $R_1 = c(t-t_1)$ and $R_2 = c(t-t_2)$. Then $R_1 - R_2 = c(t_2 - t_1)$, so the average velocity of the particle in the direction of \mathbf{r} would have to be c and that is not counting whatever velocity the charge might have in other directions. Since no charged particle can travel at the speed of light (they always have non-zero rest mass), it follows that **only one retarded point contributes to the potential, at any given moment**. Now Eqn () might suggest that the retarded potential of a point charge is simply $\frac{q}{R}$, the same as in the static case only with the understanding that R is the distance to the retarded position of the charge. But this is wrong, for a very subtle reason. It is true that for a point source the denominator R comes outside the integral, but what remains ,

$$\int \rho(\mathbf{r}', t_{ret}) d^3 \mathbf{r}' \quad (28)$$

is **not** equal to the charge of the particle. To calculate the total charge of a configuration one must integrate ρ over the entire distribution at *one instant of time*, but here the retardation obliges us to evaluate $\rho(\mathbf{r}', t_{ret})$ at *different times* for different parts of the configuration. If the source is moving, this will give a distorted picture of the total charge configuration. Apparently, this problem should disappear for *point charges*, but it doesn't. In Maxwell's electrodynamics, formulated as it is in terms of charge and current *densities*, a point charge must be regarded as the limit of an extended charge, when the size goes to zero. And for an extended particle, no matter how small, the retardation in (28) throws in a factor $\frac{1}{(1 - \hat{\mathbf{R}} \cdot \frac{\mathbf{v}}{c})}$, where \mathbf{v} is the velocity of the charge *at the retarded time* and $\hat{\mathbf{R}}$ is the unit vector along \mathbf{R} . Therefore

$$\int \rho(\mathbf{r}', t_{ret}) d^3 \mathbf{r}' = \frac{q}{1 - \hat{\mathbf{R}} \cdot \frac{\mathbf{v}}{c}} \quad (29)$$

To prove this, consider an object moving in a straight line with a speed v towards an observer on that line. The dimension of the object along that line is L (L is *not the rest length*, but the length as measured by the observer). The object will look longer than L , because the light one receives from the end left earlier than the light received *simultaneously* from the front, and at that earlier time the object was farther away (see figure). In the interval it takes light from the end to travel the extra distance L' , the object itself moves a distance $L' - L$: $\frac{L'}{c} = \frac{L' - L}{v}$, or $L' = \frac{L}{1 - \frac{v}{c}}$. So the approaching object appears longer by a factor $\frac{1}{1 - \frac{v}{c}}$. By contrast, an object going away from the observer looks shorter, by a factor $\frac{1}{1 + \frac{v}{c}}$. In general, if the object's velocity makes an angle θ with the line of sight, the extra distance light from the end must cover is $L' \cos \theta$. Therefore, $\frac{L' \cos \theta}{c} = \frac{L' - L}{v}$ or $L' = \frac{L}{1 - \frac{v \cos \theta}{c}}$. Notice that this effect

does not distort the dimensions perpendicular to the motion. The apparent volume V' of the object, then, is related to the actual volume V by

$$V' = \frac{V}{1 - \frac{\mathbf{R} \cdot \mathbf{v}}{c}} \quad (30)$$

Therefore, whenever we do an integral of the type (30), in which the integrand is evaluated at the retarded time, the effective volume is modified by the factor in (30), just as the apparent volume of the train was, and for the same reason. Because this correction factor makes no reference to size of the particle, it is as significant for a point charge as for an extended charge. It follows, then, that

$$\phi(\mathbf{r}, t) = \frac{qc}{Rc - \mathbf{R} \cdot \mathbf{v}} \quad (31)$$

where \mathbf{v} is the velocity of the charge *at the retarded time* and \mathbf{R} is the vector from the retarded position to the field point \mathbf{r} . Similarly the vector potential is given by

$$\mathbf{A}(\mathbf{r}, t) = \frac{q\mathbf{v}}{Rc - \mathbf{R} \cdot \mathbf{v}} \quad (32)$$

Eqns (31) and (32) are the famous **Lienard-Wiechert potentials**.

5 THE FIELDS OF A MOVING POINT CHARGE

The differentiation of the potentials yield the fields, but the calculations though straightforward are lengthy. Hence we just state the results here (the interested reader may see J.D. Jackson, Sect. 14.1).

$$\mathbf{E}(\mathbf{r}, t) = \frac{qR}{(\mathbf{R} \cdot \mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \mathbf{R} \times (\mathbf{u} \times \mathbf{a})] \quad (33)$$

where $\mathbf{u} = c\hat{\mathbf{R}} - \mathbf{v}$ and \mathbf{a} is the acceleration, all quantities being evaluated at the retarded time. The magnetic field is given by

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t) \quad (34)$$

Therefore the *magnetic field of a point charge is always perpendicular to the electric field, and to the vector from the retarded point*. The first term in \mathbf{E} falls off as the inverse square of the distance from the particle. If the velocity and acceleration are *both zero* it reduces to the electrostatic case, i.e., Coulomb's law. For this reason the first term in \mathbf{E} is sometimes called the **Generalized Coulomb field**. Since it does not depend on the acceleration, it is also called the **velocity field**. The second term falls off inversely as the

distance from the particle and is therefore dominant at large distances. Since it is proportional to the acceleration it is also called the **acceleration field**. For reasons given later, this term is also called the **radiation field**. The same terminology applies to the magnetic field.

6 RADIATION

Like all electromagnetic fields the source of electromagnetic waves is some arrangement of electric charge. But a charge at rest does not generate electromagnetic waves; nor does a steady current. It takes **accelerating charges** and **changing currents** to produce em waves, i.e., to **radiate**. Once established, the em waves in vacuum propagate out to infinity, carrying energy with them; the signature of radiation is this irreversible flow of energy away from the source. Imagine a spherical shell at radius r ; the total power $P(r)$ passing out through this surface is the integral of the Poynting vector:

$$P(r) = \int \mathbf{S} \cdot d\mathbf{A} = \frac{c}{4\pi} \int (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{A} \quad (35)$$

The power radiated is the limit of this quantity as r goes to infinity:

$$P_{rad} = \lim_{r \rightarrow \infty} P(r) \quad (36)$$

This is the energy (per unit time) that is transported to infinity, and never comes back. Now, the area of the sphere is $4\pi r^2$, so for radiation to occur the Poynting vector must decrease (at large r) no faster than $\frac{1}{r^2}$. According to Coulomb's law, electrostatic fields fall off like $\frac{1}{r^2}$ (or even faster, if the total charge is zero), and the Biot-Savart law says magnetostatic fields have the same r dependence, which means that $S \sim \frac{1}{r^4}$, for static configurations. So, static sources do not radiate. The study of radiation, then, involves picking out the parts of \mathbf{E} and \mathbf{B} that go like $\frac{1}{r}$ at large distances from the source, constructing from them the $\frac{1}{r^2}$ term in \mathbf{S} , integrating over a large spherical surface, and taking the limit as $r \rightarrow \infty$.

7 RADIATION BY A POINT CHARGE

From (33) and (34), the Poynting vector is given by

$$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) = \frac{c}{4\pi} [\mathbf{E} \times (\hat{\mathbf{R}} \times \mathbf{E})] = \frac{c}{4\pi} [E^2 \hat{\mathbf{R}} - (\hat{\mathbf{R}} \cdot \mathbf{E})\mathbf{E}] \quad (37)$$

However, not all of this energy flux constitutes radiation; some of it is just field energy carried along by the particle as it moves. The *radiated* energy, in effect,

detaches itself from the charge and propagates off to infinity. To calculate the total power radiated by the particle at time t_{ret} , we draw a huge sphere of radius R centered at the retarded position of the particle, wait the appropriate interval $(t - T_{ret})$ for the radiation to reach the sphere, and at that moment integrate the Poynting vector over the surface. Now, the area of the sphere varies as r^2 , so any term in \mathbf{S} that goes like $\frac{1}{r^2}$ will yield a finite answer, but terms containing higher inverse powers of R will contribute nothing in the limit $r \rightarrow \infty$. For this reason, only the acceleration fields represent true radiation, and hence their name radiation fields. *The velocity fields carry energy, but as the charge moves, this energy is dragged along-hence its not radiation.*

$$\mathbf{S}_{rad} = \frac{c}{4\pi} E_{rad}^2 \hat{\mathbf{R}} \quad (38)$$

Going to the instantaneous rest frame of the charge $\mathbf{u} = c\hat{\mathbf{R}}$ and

$$\mathbf{E}_{rad} = \frac{q}{c^2 R} [\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \mathbf{a})] = \frac{q}{c^2 R} [(\hat{\mathbf{R}} \cdot \mathbf{a})\hat{\mathbf{R}} - \mathbf{a}] \quad (39)$$

Then

$$\mathbf{S}_{rad} = \frac{c}{4\pi} \frac{q^2}{c^4 R^2} [a^2 - (\hat{\mathbf{R}} \cdot \mathbf{a})^2] \hat{\mathbf{R}} = \frac{q^2}{4\pi c^3} \left(\frac{\sin^2 \theta}{R^2}\right) \hat{\mathbf{R}} \quad (40)$$

where θ is the angle between $\hat{\mathbf{R}}$ and \mathbf{a} . The total power radiated is evidently

$$P = \int \mathbf{S}_{rad} \cdot d\mathbf{A} = \frac{2q^2 a^2}{3c^3} \quad (41)$$

This is the **Larmor's formula**, for nonrelativistic charged particles. We see that if the particle moves with uniform velocity (i.e. $a = 0$), the total power radiated is also zero. Thus only accelerating charges can radiate.

8 REFERENCES

Jackson, Landau and Lifshitz Griffiths Rybicki and Lightman