Methods of Mathematical Physics-I

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Second order Partial Differential Equations

Very commonly encountered in Physics

Some examples

Laplace Equation: $\nabla^2\psi = 0$

Poisson Equation: $\nabla^2\psi = 4\pi\rho$

Wave Equation: $\nabla^2\psi + k^2\psi = 0$

Diffusion Equation: $\nabla^2\psi = \frac{1}{a^2}\frac{\partial\psi}{\partial t}$

Schrödinger Equation: $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$

Consider a second order PDE in two variables described by an operator of the form

$$\mathcal{L} = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} + f$$

Based on the 2nd order terms, the equation $\mathcal{L}y = 0$ is called

Elliptic if $ac - b^2 > 0$

Parabolic if $ac - b^2 = 0$

Hyperbolic if $ac - b^2 < 0$

Elliptic operators may be cast in the form $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$ or just ∇^2

Parabolic in the form $a\frac{\partial}{\partial t} + \nabla^2$ and Hyperbolic as $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

$$\mathcal{L} = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2}$$ may be factorised as

$$\mathcal{L} = \left(\frac{b + \sqrt{b^2 - ac}}{c^{1/2}}\frac{\partial}{\partial x} + c^{1/2}\frac{\partial}{\partial y}\right)\left(\frac{b - \sqrt{b^2 - ac}}{c^{1/2}}\frac{\partial}{\partial x} + c^{1/2}\frac{\partial}{\partial y}\right)$$

This is a product of two first order partial differential operators, each would have their own characteristics.

So $\mathcal{L}$ would possess two sets of characteristics with slope

$$\frac{dy}{dx} = \frac{c}{b \pm \sqrt{b^2 - ac}}$$

For a Parabolic operator, the two sets merge into one

A Hyperbolic operator has real characteristics

Characteristics of an Elliptic operator are imaginary and thus not useful for propagating boundary conditions in real space

Separation of variables

This is one of many ways of converting a PDE into a set of ODEs

Example: Helmholtz equation in Cartesian coordinates:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0$$

Assume that $\psi(x, y, z) = X(x)Y(y)Z(z)$

Then $YZ\frac{\partial^2 X}{\partial x^2} + ZX\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} + k^2 XYZ = 0$

Hence $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + k^2 = 0$

So $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - k^2 = \text{const} = -l^2$

Function of *x* alone

Function of *y,z* alone

Constant of separation

Similarly $\quad \dfrac{1}{Y}\dfrac{\partial^2 Y}{\partial y^2} = -\dfrac{1}{Z}\dfrac{\partial^2 Z}{\partial z^2} - k^2 + l^2 = -m^2$

and $\quad \dfrac{1}{Z}\dfrac{\partial^2 Z}{\partial z^2} = l^2 + m^2 - k^2 = -n^2\,, \quad$ i.e. $\quad k^2 = l^2 + m^2 + n^2$

Thus we have three ODEs of the form

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -l^2, \text{ i.e. } \frac{d^2 X}{dx^2} + l^2 X = 0$$

We can consider this to be a Sturm-Liouville form $(pX')' + qX = \lambda X$

with $p = 1,\ q = 0$ and eigenvalue $\lambda = -l^2$, weight $w(x) = 1$

Periodic boundary conditions $X(0) = X(2\pi),\ X'(0) = X'(2\pi)$

would then render the operator Hermitian, with orthogonal set of eigenfunctions

$X_l = \mathrm{e}^{\pm ilx}$ or $X_l = \cos(lx),\ \sin(lx)$

and the eigenvalue l taking integer values in the range $[0, \infty]$

Clearly $\int_0^{2\pi} X_{l_1}^* X_{l_2} \, dx = 0 \quad$ for $\quad l_1 \neq l_2$

and $\int_0^{2\pi} X_l^* X_l \, dx = 2\pi$ (exponential form) or π (sine/cosine form)

The orthonormal eigenfunctions are

$$\phi_l(x) = \frac{\mathrm{e}^{ilx}}{\sqrt{2\pi}} \qquad (l = ... -2, -1, 0, 1, 2, ...)$$

or $$\frac{\cos(lx)}{\sqrt{\pi}}, \; \frac{\sin(lx)}{\sqrt{\pi}} \qquad (l = 0, 1, 2, ...)$$

such that $\int_0^{2\pi} \phi_j^* \phi_l \, dx = \delta_{jl}$