Methods of Mathematical Physics-I

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Linear function spaces (Vector spaces)

Let $\phi_1(s), \phi_2(s), \phi_3(s), ..., \phi_n(s)$ be a set of linearly independent functions

These can be used as *basis functions* to construct a linear function space of which each linear combination of these basis functions

$$f(s) = \sum_{i=1}^{n} a_i \phi_i(s)$$ is a member.

a_i are coefficients independent of s
They may be real or complex
Equivalent to coordinates in function space

If $g(s) = \sum_{i=1}^{n} b_i \phi_i(s)$ is another member of the function space, then

$$h(s) = f(s) + g(s) = \sum_{i=1}^{n} (a_i + b_i)\phi_i(s)$$ is also a member
[closed under addition]

$$u(s) = kf(s) = \sum_{i=1}^{n} k a_i \phi_i(s)$$ (k a scalar) is also a member
[closed under multiplication]

Linear function spaces (Vector spaces)

Scalar Product $$\langle f|g\rangle = \int_a^b f^*(s)g(s)w(s)ds$$

with *a*, *b*, *w*(*s*) being chosen as a part of the definition of the scalar product. *w*(*s*) must be positive definite in the interval [*a*,*b*].

If a function space is closed under addition and multiplication by a scalar and if a scalar product exists for all pairs of its members then such a function space is called a *Hilbert Space*.

If the basis functions are such that $\langle \phi_i|\phi_j\rangle = 0$ for $i \neq j$ then they are called *orthogonal*. In addition if $\langle \phi_i|\phi_i\rangle = 1$ for all *i* then the basis functions are called *orthonormal*.

In Hilbert space the following inequalities hold:

Schwartz Inequality: $|\langle f|g\rangle|^2 \leq \langle f|f\rangle\langle g|g\rangle$

Bessel's Inequality: $\langle f|f\rangle \geq \sum_{i=1}^{n} |a_i|^2$ if bases are orthonormal

Linear operators in Hilbert Space: $\mathcal{A}f = \bar{f}$ (another function in the same space)

$$\mathcal{A}(f+g) = \mathcal{A}f + \mathcal{A}g; \;\; (\mathcal{A}+\mathcal{B})f = \mathcal{A}f + \mathcal{B}f; \;\; \mathcal{A}k = k\mathcal{A}$$

If inverse exists then $\mathcal{A}^{-1}\mathcal{A} = 1 = \mathcal{A}\mathcal{A}^{-1}$

Adjoint $\mathcal{A}^\dagger$: $\langle f|\mathcal{A}g\rangle = \langle \mathcal{A}^\dagger f|g\rangle$

If $\mathcal{A} = \mathcal{A}^\dagger$ then the operator is Self-adjoint, also called *Hermitian*.

Hermitian operators are important in physics because they have real eigenvalues, representing measurable physical quantities.

If $\mathcal{A}^\dagger = \mathcal{A}^{-1}$ then the operator is called *Unitary*.

An operator that is Real and Unitary is called *Orthogonal*.

Eigenvalues and Eigenfunctions

Example: Schrödinger's Equation $\mathcal{H}\psi = E\psi$

$\mathcal{H}$ is a differential operator, E is a scalar

E is a fixed number for which a solution is sought.

Solution may not exist for all values of E

Values of E that admit solutions are called *Eigenvalues*.

The corresponding solutions ψ are called *Eigenfunctions*.

In general $\mathcal{L}u(x) + \lambda w(x)u(x) = 0$

where λ is the eigenvalue and

$w(x)$ is a weight function that appears in the definition of scalar product

For a given λ, the function $u_\lambda(x)$ that satisfies the equation along with the given boundary conditions is the corresponding eigenfunction

Existence of eigenfunction is not guaranteed for an arbitrary λ.

Often the eigenvalues are discrete, dictated by the boundary conditions.

Sturm-Liouville Theory

Consider a second order linear differential operator

$$\mathcal{L} = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$$

If this can be cast in the form

$$\mathcal{L}u = \frac{d}{dx}\left[p_0(x)\frac{du}{dx}\right] + p_2(x)u(x)$$

Then $\mathcal{L}$ is called a *self-adjoint* or *Sturm-Liouville* operator

This requires $$p_0'(x) = p_1(x)$$

If the original operator is not self-adjoint then it can be rendered so by multiplying a weight function

$$w(x) = \frac{1}{p_0(x)}\exp\left[\int \frac{p_1(x)}{p_0(x)}dx\right]$$

This is possible if the zeros of $p_0(x)$ do not lie within the domain of interest

Rewrite the self-adjoint operator as

$$\mathcal{L}u = \frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u(x) = [pu']' + qu$$

$\mathcal{L}u = f(x)$ would admit two linearly independent solutions. Let's call them $u(x)$ and $v(x)$. The boundary conditions will decide the linear combination that will be the desired solution.

If $x = [a, b]$ define the boundary, then the boundary conditions could be

- Dirichlet: on $u(a), v(a); \;\; u(b), v(b)$
- Neumann: on $u'(a), v'(a); \;\; u'(b), v'(b)$

The scalar product $\langle v|\mathcal{L}u\rangle = \int_a^b v^* \mathcal{L}u \, dx = \int_a^b v^* (pu')' \, dx + \int_a^b v^* qu \, dx$

The 1st term: $v^* pu'|_a^b - \int_a^b v^{*\prime} pu' \, dx = v^* pu'|_a^b - v^{*\prime} pu|_a^b + \int_a^b u(pv^{*\prime})' \, dx$

So $\langle v|\mathcal{L}u\rangle = [v^* pu' - v^{*\prime} pu]_a^b + \int_a^b [(pv^{*\prime})' + qv^*]u \, dx$

If such boundary conditions are specified that $[v^* p u' - v^{*\prime} p u]_a^b = 0$

then $\langle v|\mathcal{L}u\rangle = \int_a^b v^* \mathcal{L}u \, dx = \int_a^b (\mathcal{L}v)^* u \, dx = \langle \mathcal{L}v|u\rangle$

Thus $\mathcal{L}$ is self-adjoint.

Boundary conditions of the type

$u, v = 0$ at the boundaries (Dirichlet)

$u', v' = 0$ at the boundaries (Neumann)

$v^* p u'|_a = v^* p u'|_b$ for all u, v (Periodic)

Will all satisfy the requirement of boundary conditions for self-adjoincy

Note that any linear combination of functions satisfying the above boundary conditions will also satisfy the same.