Methods of Mathematical Physics-I

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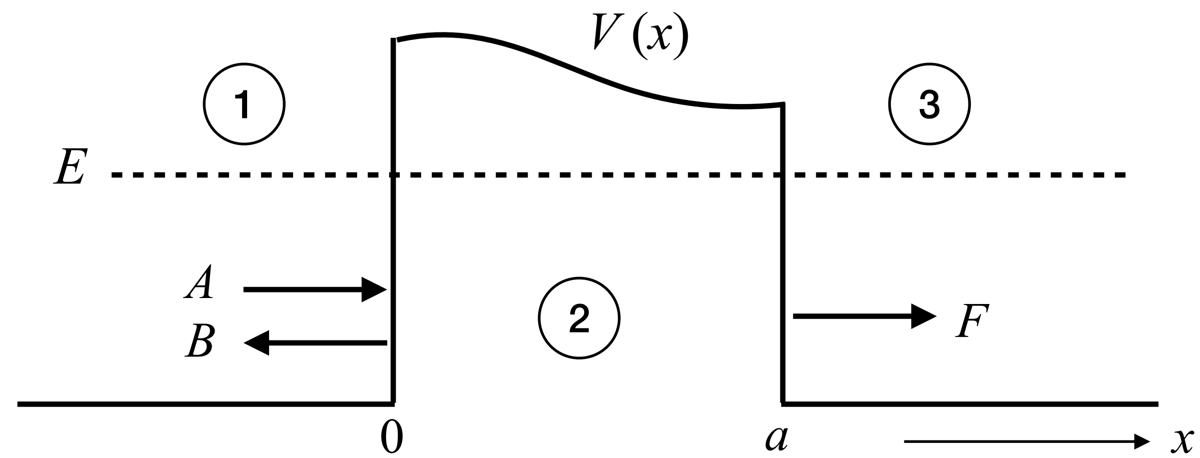
WKB examples

Tunnelling



$$\psi(x) = A\mathrm{e}^{ikx} + B\mathrm{e}^{-ikx}, \quad x < 0 \qquad \text{Region } 1$$

$$\psi(x) = \frac{C}{\sqrt{\kappa(x)}}\mathrm{e}^{\int_0^x \kappa(t)dt} + \frac{D}{\sqrt{\kappa(x)}}\mathrm{e}^{-\int_0^x \kappa(t)dt}, \quad 0 < x < a \qquad \text{Region } 2$$

$$\psi(x) = F\mathrm{e}^{ikx}, \quad x > a \qquad \text{Region } 3$$

Wave propagating in $+x$ direction is incident from $x<0$ in region 1. This is represented by term A. Term B is the reflected component.

In region 2, term C increases exponentially and term D decreases with increasing x. However as $E<V$ in this region, the wave function should attenuate.

Thus $C \ll D$. Approximate $C \longrightarrow 0$. (The higher the V, the smaller the value of C)

Then $\psi(x) \approx \dfrac{D}{\sqrt{\kappa(x)}}\mathrm{e}^{-\int_0^x \kappa(t)dt}$ in region 2

Thus tunnelling (transmission) probability

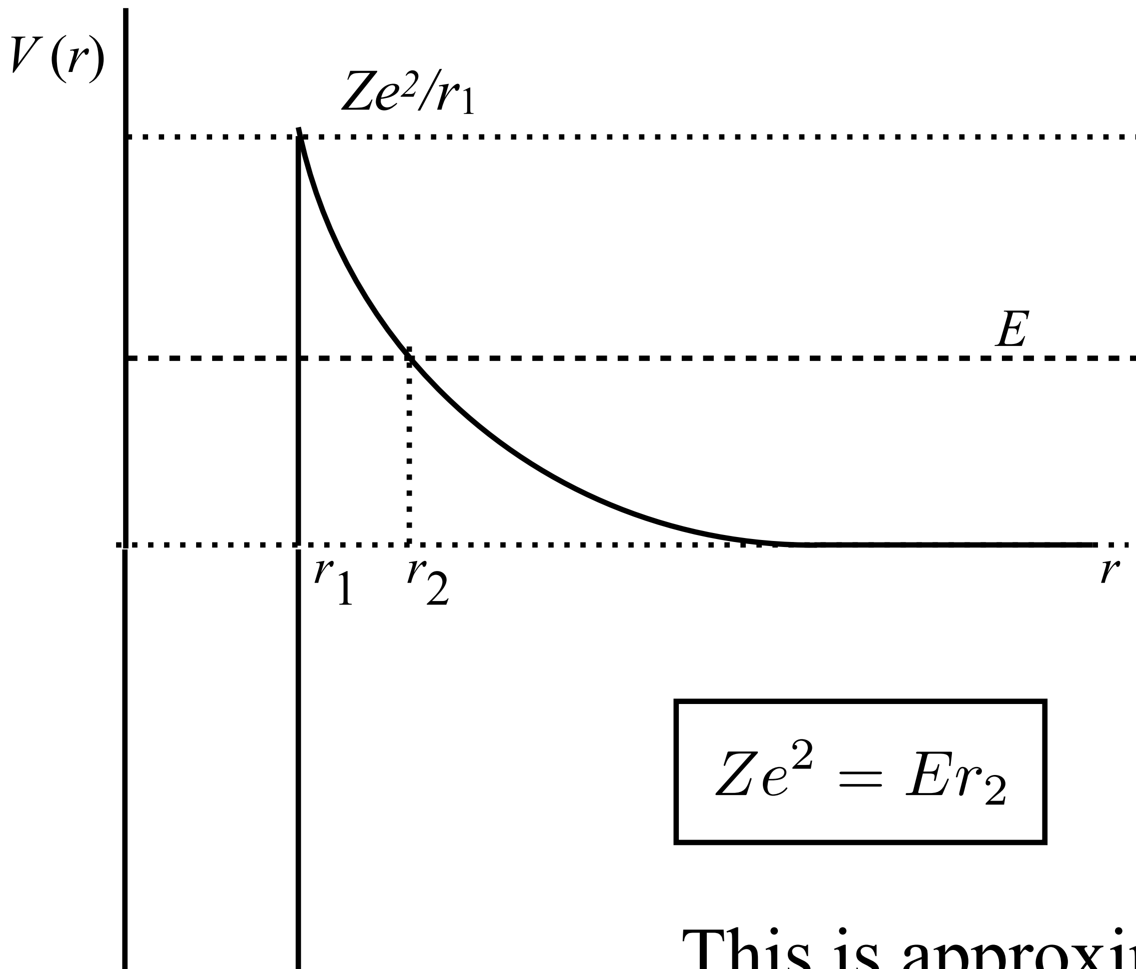
$$T = \frac{|\psi(x=a)|^2}{|\psi(x=0)|^2} = \frac{\frac{D^2}{\kappa(a)} e^{-2\int_0^a \kappa(t)dt}}{\frac{D^2}{\kappa(0)}}$$

$$= \frac{\kappa(0)}{\kappa(a)} \exp\left[-2\int_0^a \kappa(t)dt\right]$$

Similar treatment for negative *x*-going wave would show that

$$T = \frac{\kappa(a)}{\kappa(0)} \exp\left[-2\int_a^0 \kappa(t)dt\right]$$

Fusion: Tunnelling through Coulomb barrier



$$Ze^2 = Er_2$$

Here $\kappa(r) = \frac{1}{\hbar}\sqrt{2m\left(\frac{Ze^2}{r} - E\right)}$

and

$$T \approx \left[\sqrt{\frac{\frac{Ze^2}{r_2} - E}{\frac{Ze^2}{r_1} - E}}\right] \times$$

$$\exp\left[-\frac{2}{\hbar}\int_{r_2}^{r_1}\sqrt{2m\left(\frac{Ze^2}{r} - E\right)}dr\right]$$

This is approximate because the rise of potential at r_2 is gradual.

So $$T \approx A\exp\left[-\frac{2\sqrt{2mE}}{\hbar}\int_{r_2}^{r_1}\sqrt{\frac{r_2}{r} - 1}\,dr\right]$$

$$= A\exp\left[-\frac{2\sqrt{2mE}}{\hbar}r_2\int_{1}^{r_1/r_2}\sqrt{\frac{1}{x} - 1}\,dx\right]$$

$$x = \frac{r}{r_2}$$

Hence

$$T \approx A \exp\left[-\frac{2\sqrt{2mE}}{\hbar}\frac{Ze^2}{E}\int_1^{r_1/r_2}\sqrt{\frac{1}{x}-1}\,dx\right]$$

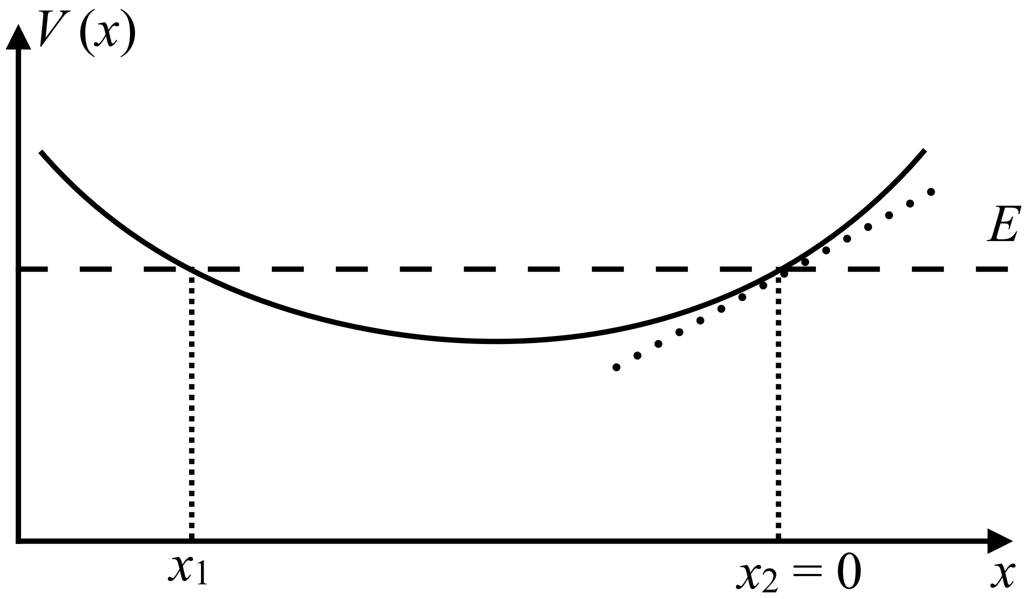
$$= A \exp\left[-K_1 \frac{Z}{\sqrt{E}}\alpha\right]$$

Note that $r=r_2$ is a classical "turning point".

WKB approximation is not a good one near the turning points as $k(x)$ or $\kappa(x)$ go to zero, and $1/\sqrt{k(x)}$ or $1/\sqrt{\kappa(x)}$ diverge.

So a patching zone is introduced to connect WKB solutions away from the turning points

Behaviour near classical turning points



Consider linear approximation of the potential near the turning point

Let us choose the origin at the turning point x_2 and write $V(x)$-E as

$$V(x) - E = V'(0) \cdot x + O(x^2)$$

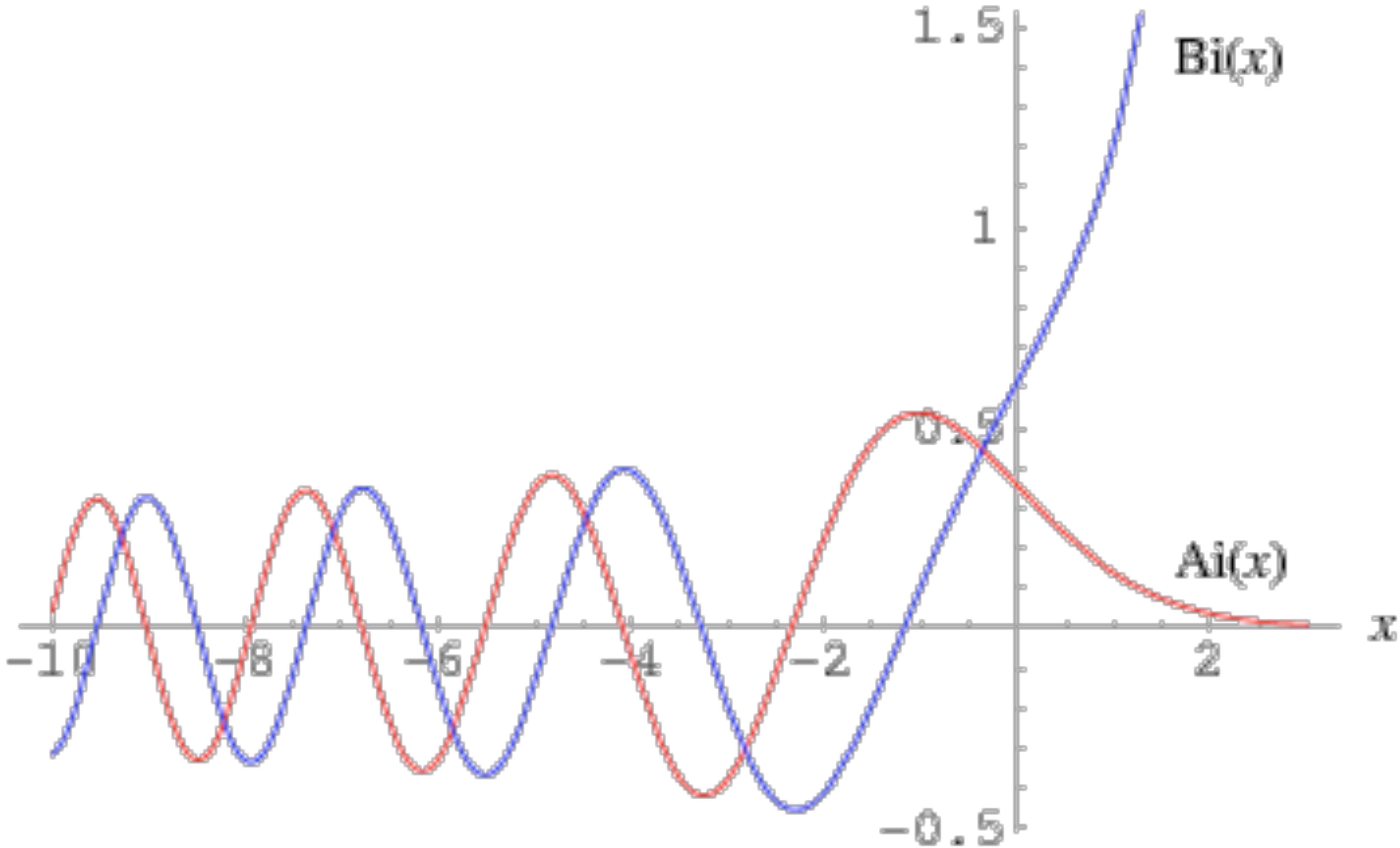
For this we would have $$\frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2} V'(0) x \psi = 0$$

write this as $$\frac{d^2\psi}{\alpha^2 dx^2} - \alpha x \psi = 0$$ where $$\alpha^3 = \frac{2m}{\hbar^2} V'(0)$$

or $$\frac{d^2\psi}{dz^2} - z\psi = 0$$ with $z = \alpha x$

This is Airy's equation with solutions Ai(x) and Bi(x). Near the turning points they represent the solution with good accuracy. The task is now to match these functions with WKB solutions away from the turning points.

Airy Functions



Wolfram Mathworld

Away from the turning points,

$$k(x) = \frac{2m}{\hbar^2}\left[E - V(x)\right]^{1/2} = \frac{2m}{\hbar^2}\left[-V'(0)x\right]^{1/2} = \alpha^{3/2}(-x)^{1/2}, \quad x < 0$$

and $\kappa(x) = \alpha^{3/2}x^{1/2}, \quad x > 0$

giving

$$\psi_{\text{WKB}} = \frac{1}{\alpha^{3/4}(-x)^{1/4}}\left\{B\exp\left[i\frac{2}{3}(-\alpha x)^{3/2}\right] + C\exp\left[-i\frac{2}{3}(-\alpha x)^{3/2}\right]\right\}, \qquad x < 0$$

$$\psi_{\text{WKB}} = \frac{1}{\alpha^{3/4}x^{1/4}}\left\{D\exp\left[-\frac{2}{3}(\alpha x)^{3/2}\right]\right\}, \qquad x > 0$$

Now, Airy functions have the asymptotic form

drop the exponentially growing term

$$\psi_{\text{Airy}}(x) \approx \frac{a}{2\sqrt{\pi}(\alpha x)^{1/4}}\exp\left[-\frac{2}{3}(\alpha x)^{3/2}\right] + \frac{b}{2\sqrt{\pi}(\alpha x)^{1/4}}\exp\left[\frac{2}{3}(\alpha x)^{3/2}\right], \qquad x \gg 0$$

$$\psi_{\text{Airy}}(x) \approx \frac{a}{\sqrt{\pi}(-\alpha x)^{1/4}}\cdot\frac{1}{2i}\left\{\exp\left[i\frac{\pi}{4} + i\frac{2}{3}(-\alpha x)^{3/2}\right] - \exp\left[-i\frac{\pi}{4} - i\frac{2}{3}(-\alpha x)^{3/2}\right]\right\},$$

$$x \ll 0$$

Comparing coefficients,

$$\frac{B}{\sqrt{\alpha}} = \frac{a}{2i\sqrt{\pi}} e^{i\pi/4} \quad ; \quad \frac{C}{\sqrt{\alpha}} = -\frac{a}{2i\sqrt{\pi}} e^{-i\pi/4} \quad ; \quad \frac{D}{\sqrt{\alpha}} = \frac{a}{2\sqrt{\pi}}$$

Hence $\quad B = -ie^{i\pi/4} D \quad ; \quad C = +ie^{-i\pi/4} D$

Using this, the Airy connections are then eliminated, the WKB coefficients are in direct relation to each other