Methods of Mathematical Physics-I

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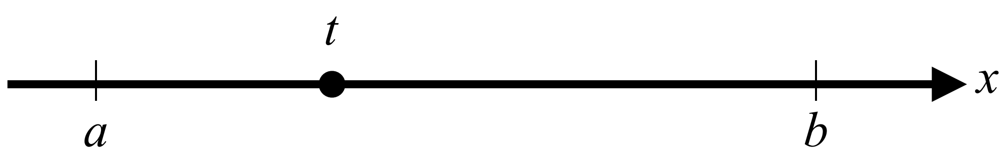
Form of Green's Function $G(x,t)$:



Homogeneous Boundary conditions at $x = a, b$: satisfied by $G(x, t)$

Let $y_1(x)$ be a solution of $\mathcal{L}y = 0$ that satisfies the B.C. at $x=a$
and $y_2(x)$ be a solution of $\mathcal{L}y = 0$ that satisfies the B.C. at $x=b$, $a < b$

Then
$$G(x,t) = y_1(x)h_1(t), \quad a \le x < t$$
$$= y_2(x)h_2(t), \quad t < x \le b$$

h_1, h_2 to be determined

We note that $G(x, t)$ is continuous at $x = t$, so $\dfrac{h_1(t)}{y_2(t)} = \dfrac{h_2(t)}{y_1(t)} = C$

and $G(x, t) = G^*(t, x)$, giving

$$y_1(x)h_1(t) = y_2^*(t)h_2^*(x), \;\; x < t$$
$$y_2(x)h_2(t) = y_1^*(t)h_1^*(x), \;\; x > t$$

Assuming that y_1 and y_2 may be chosen to be real,

$$h_2^*(x) = Cy_1(x), \; h_1^*(x) = Cy_2(x)$$

and thus
$$h_2(x) = Ay_1(x), \; h_1(x) = Ay_2(x) \qquad (A \equiv C^*)$$

Using this, $$G(x,t) = Ay_1(x)y_2(t), \quad x < t$$
$$= Ay_2(x)y_1(t), \quad x > t$$

Thus $\left.\dfrac{dG}{dx}\right|_{t+} = Ay_2'(t)y_1(t)$ and $\left.\dfrac{dG}{dx}\right|_{t-} = Ay_1'(t)y_2(t)$

So $A[y_2'(t)y_1(t) - y_1'(t)y_2(t)] = \dfrac{1}{p(t)}$

i.e. $A = [p(t)\{y_2'(t)y_1(t) - y_1'(t)y_2(t)\}]^{-1} = [p(t)W(y_1, y_2)]^{-1}$

Now we note

$\mathcal{L}$ is self-adjoint, $\mathcal{L}y = (py')' + qy$ and $\mathcal{L}y = \lambda y$

$y_1,\ y_2$ are two solutions for the same λ

So $(\mathcal{L}y_1)y_2 - y_1(\mathcal{L}y_2) = (py_1')'y_2 + qy_1y_2 - y_1(py_2')' - qy_1y_2$

$$= (py_1')'y_2 - (py_2')'y_1$$

$$\text{Or} \quad (\lambda y_1)y_2 - y_1(\lambda y_2) = (py_1')'y_2 + py_1'y_2' - py_2'y_1' - (py_2')'y_1$$

$$= \frac{d}{dx}[(py_1')y_2] - \frac{d}{dx}[(py_2')y_1]$$

$$= \frac{d}{dx}[(py_1')y_2 - (py_2')y_1]$$

$$= \frac{d}{dx}[p\{y_1'y_2 - y_2'y_1\}] = -\frac{d}{dx}[pW(y_1, y_2)]$$

Since the LHS = $\lambda y_1 y_2 - y_1 \lambda y_2 = 0$,

We have $\quad \dfrac{d}{dx}[pW(y_1, y_2)] = 0\,,$ $\quad$ i.e. $\quad pW(y_1, y_2)$= const.

Hence A is independent of t .

The results so far are derived for homogeneous Boundary Conditions.

In other cases, it may be possible to render the boundary conditions homogeneous by a change of the dependent variable.

For example, if the specified B.C. are

$$y = C_1 \text{ at } x = a \text{ and } y = C_2 \text{ at } x = b$$

then a change of the dependent variable to

$$u = y - \frac{C_1(b-x) + C_2(x-a)}{(b-a)}$$

will make the B.C. homogeneous: $u = 0 \text{ at } x = a, b$

The ODE can then be re-cast in terms of u

The symmetry of $G(x,t)$ discussed so far w.r.t. the two end points arises from the B.C. being specified at the two ends. For other types of B.C. this symmetry will not be present.

Example: an Initial Value Problem

Consider $\mathcal{L}y = \dfrac{d^2y}{dx^2} + y = f(x), \quad x \in [0, \infty]$ with B.C. $y = y' = 0$ at $x = 0$

$\mathcal{L}y = 0$ has two solutions: $\sin(x)$ and $\cos(x)$

The only combination of these that satisfies the B.C. at $x = 0$ is $y = 0$

So for $x < t$, $G(x, t) = 0$

At $x > t$, No B.C. are available. So we may write

$$G(x,t) = c_1(t)y_1(x) + c_2(t)y_2(x) = c_1(t)\sin(x) + c_2(t)\cos(x)$$

Continuity of G at $x = t$ demands $0 = c_1(t)\sin(t) + c_2(t)\cos(t)$

And the derivative condition $\left.\dfrac{dG}{dx}\right|_{t+} - \left.\dfrac{dG}{dx}\right|_{t-} = \dfrac{1}{p(t)} = 1$

implies that $c_1(t)\cos(t) - c_2(t)\sin(t) - 0 = 1$

Hence $c_1(t) = \cos(t), \quad c_2(t) = -\sin(t)$

So $G(x,t) = \cos(t)\sin(x) - \sin(t)\cos(x) = \sin(x-t), \quad x > t$

$$= 0, \quad x < t$$

Thus
$$y(x) = \int_0^\infty G(x,t) f(t)\, dt$$

$$= \int_0^x \sin(x-t) f(t)\, dt$$

$y(x)$ depends only on the past history of the function $f(t)$

Boundary conditions at infinity

Example: Consider Helmholtz's equation

$$\mathcal{L}\psi(x) = \left(\frac{d^2}{dx^2} + k^2\right)\psi(x) = g(x)$$

$\mathcal{L}\psi(x) = 0$ has solutions of the form $\mathrm{e}^{\pm ikx}$

Impose Outgoing Wave boundary conditions:

$$\text{at } x = +\infty, \quad \psi(x) = y_2(x) = \mathrm{e}^{+ikx}$$
$$\text{at } x = -\infty, \quad \psi(x) = y_1(x) = \mathrm{e}^{-ikx}$$

These are homogeneous B.C. and the operator is Hermitian with $p(x) = 1$

The Green's function may then be constructed as

$$G(x,x') = Ay_1(x)y_2(x'), \quad x < x'$$
$$= Ay_2(x)y_1(x'), \quad x > x'$$

And

$$A = \frac{1}{W(y_1,y_2)} = \frac{1}{ik+ik} = -\frac{i}{2k}$$

Thus

$$G(x,x') = -\frac{i}{2k}\mathrm{e}^{-ik(x-x')}, \quad x < x'$$
$$= -\frac{i}{2k}\mathrm{e}^{ik(x-x')}, \quad x > x'$$

or

$$G(x,x') = -\frac{i}{2k}\mathrm{e}^{-ik|x-x'|}$$