Methods of Mathematical Physics-I

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Green's Function

Green's function is often utilised to obtain solutions of inhomogeneous Differential Equations

$$\mathcal{L}y(x) = f(x) \text{ with Boundary Conditions at } x = a, b$$

Green's function $G(x,t)$ provides a solution of the form

$$y(x) = \int_a^b G(x,t) f(t)\, dt$$

Where *t* is called the Source Point and *x* the Field Point.

Let us consider a second order self-adjoint ODE

$$\mathcal{L}y = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = f(x)$$

to be satisfied in the range $a \le x \le b$ subject to homogeneous* B.C. at $x=a$ and $x=b$ that renders $\mathcal{L}$ Hermitian.

** B.C. that continue to be satisfied if the function is scaled by an arbitrary factor*

Equation satisfied by the Green's Function is

$$\mathcal{L}G(x,t) = \delta(x-t) \quad \text{along with the B.C.}$$

Because

$$\mathcal{L}y(x) = \mathcal{L}\int_a^b G(x,t)f(t)\,dt = \int_a^b [\mathcal{L}G(x,t)]f(t)\,dt$$

$$= \int_a^b \delta(x-t)f(t)\,dt = f(x)$$

Then

$$\int\limits_{t-\epsilon}^{t+\epsilon} \frac{d}{dx}\left[p(x)\frac{dG}{dx}\right]\,dx + \int\limits_{t-\epsilon}^{t+\epsilon} q(x)G(x,t)\,dx = \int\limits_{t-\epsilon}^{t+\epsilon} \delta(x-t)\,dx = 1$$

So $$p(x)\frac{dG}{dx}\bigg|_{t+\epsilon} - p(x)\frac{dG}{dx}\bigg|_{t-\epsilon} + \int\limits_{t-\epsilon}^{t+\epsilon} q(x)G(x,t)\,dx = 1$$

This cannot be satisfied if both G and G' are continuous at $\epsilon = 0$

We choose G' to be discontinuous while G is continuous at $x = t$.

Then
$$\lim_{\epsilon \to 0} \left[G'|_{t+\epsilon} - G'|_{t-\epsilon} \right] = \frac{1}{p(t)}$$

We now expand $G(x,t)$ in the eigenfunctions of $\mathcal{L}$ with the applied B.C.

$$\mathcal{L}\phi_n(x) = \lambda_n \phi_n(x) \ ; \qquad \langle \phi_n | \phi_m \rangle = \delta_{nm}$$

Now
$$G(x,t) = \sum_{n,m} g_{nm} \phi_n(x) \phi_m^*(t)$$

While
$$\delta(x-t) = \sum_m \phi_m(x) \phi_m^*(t)$$

$$\delta(x-t) = \sum_m c_m(t) \phi_m(x)$$

$$c_m(t) = \int_a^b \phi_m^*(x) \delta(x-t) dx$$
$$= \phi_m^*(t)$$

Then
$$\mathcal{L} \sum_{n,m} g_{nm} \phi_n(x) \phi_m^*(t) = \sum_m \phi_m(x) \phi_m^*(t)$$

So
$$\sum_{n,m} \lambda_n g_{nm} \phi_n(x) \phi_m^*(t) = \sum_m \phi_m(x) \phi_m^*(t)$$

Hence $$\sum_n \lambda_n \phi_n(x) \sum_m g_{nm} \phi_m^*(t) = \sum_m \phi_m(x) \phi_m^*(t)$$

Multiplying by $\phi_n^*(x)$ and integrating over x from a to b,

$$\sum_m \lambda_n g_{nm} \phi_m^*(t) = \sum_m \delta_{nm} \phi_m^*(t)$$

Taking scalar product with $\phi_m(t)$,

$$\lambda_n g_{nm} = \delta_{nm} \quad , \qquad \text{i.e.} \qquad g_{nm} = \frac{\delta_{nm}}{\lambda_n}$$

Therefore

$$G(x,t) = \sum_n \frac{\phi_n^*(t) \phi_n(x)}{\lambda_n}$$

Which reveals that

$$G(x,t) = G^*(t,x)$$