

Some solutions of MHD equations

↳ Comment

1. Eqn of isothermal MHD

$$\rho [\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}] = \nu \nabla^2 \mathbf{u} - \nabla p + \frac{\mathbf{J} \times \mathbf{B}}{\mu_0}$$

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$



$$\partial_t \mathbf{B} + (\mathbf{u} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u} - (\nabla \cdot \mathbf{u}) \mathbf{B}$$

$$p = c_s^2 \rho$$

• comments

1. non-linearity
2. Existence of solutions.
3. Hydrodynamic limit, and complexity of solutions, connection with turbulence.

$$\left. \begin{aligned} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= \nu \nabla^2 \mathbf{u} - \nabla p \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \right\}$$

4. Turbulence is ubiquitous.

2. How to solve eqns of MHD.

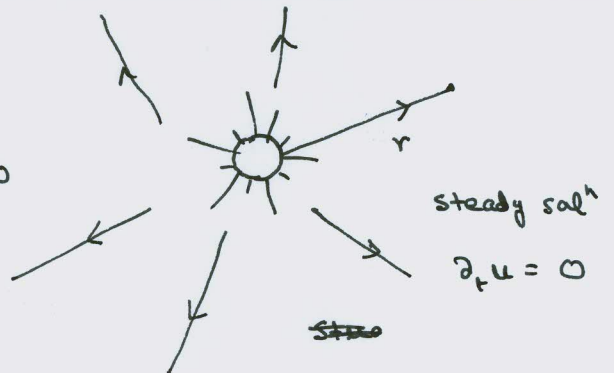
Step 1. Look for symmetry, $\vec{B} = 0$,

$$u_r = u, u_\theta = u_\phi = 0, \partial_\theta = 0, \partial_\phi = 0$$

$$\Rightarrow \rho u \frac{du}{dr} = -\frac{dp}{dr} - \frac{GM\rho}{r^2}$$

and $\frac{1}{r^2} \frac{d}{dr} (r^2 \rho u) = 0$

with $p = c_s^2 \rho \rightarrow$ isothermal

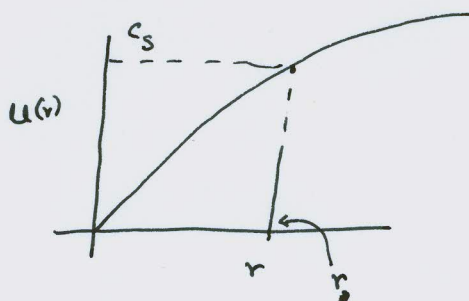


Integrate to obtain: $\frac{1}{2} u^2 + c_s^2 \ln \rho - \frac{GM}{r} = \text{constant} \equiv \mathcal{E} \leftarrow \text{energy}$

$4\pi r^2 \rho u = \text{constant} = \dot{m} \leftarrow \text{mass flux}$

Problem solved! To obtain u , eliminate ρ , to obtain:

$$\frac{1}{2} u^2 - c_s^2 \ln u - 2c_s^2 \ln r - \frac{GM}{r} = \mathcal{E}', \quad \mathcal{E}' = \mathcal{E} - c_s^2 \ln \left(\frac{\dot{m}}{4\pi} \right)$$



$$c_s = \left(\frac{GM}{2r_s} \right)^{1/2}$$

$$r_s \sim 5R_\odot$$

Parker Wind

3. How to solve eqns of MHD

step 2 stability of the solution.

$$\rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = \nu \nabla^2 \mathbf{u} - \nabla P + \frac{\mathbf{J} \times \mathbf{B}}{\mu_0}$$

magnetic pressure

$\swarrow -\frac{1}{2} \nabla B^2$
 $\searrow \mathbf{B} \cdot \nabla \mathbf{B}$

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\partial_t \mathbf{B} + (\mathbf{u} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u} - (\nabla \cdot \mathbf{u}) \mathbf{B} + \eta \nabla^2 \mathbf{B}$$

$$\mathbf{u} \rightarrow \mathbf{u}_0 + \delta \mathbf{u}; \quad \mathbf{B} \rightarrow \mathbf{B}_0 + \delta \mathbf{b}, \quad \rho \rightarrow \rho_0 + \delta \rho$$

Substitute and expand:

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = (\mathbf{u}_0 + \delta \mathbf{u}) \cdot \nabla (\mathbf{u}_0 + \delta \mathbf{u})$$

$$= (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 + (\delta \mathbf{u} \cdot \nabla) \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla) \delta \mathbf{u} + (\delta \mathbf{u} \cdot \nabla) \delta \mathbf{u}$$

Ignore second order terms (infinitesimal perturbation)

to obtain:

$$\rho_0 \partial_t \delta \mathbf{u} = -c_s^2 \nabla \delta \rho - \nabla \left(\frac{\mathbf{B}_0 \cdot \delta \mathbf{b}}{\mu_0} \right) + \frac{1}{\mu_0} (\mathbf{B}_0 \cdot \nabla) \delta \mathbf{b}$$

$$\partial_t \delta \mathbf{b} = (\mathbf{B}_0 \cdot \nabla) \delta \mathbf{u} - (\nabla \cdot \delta \mathbf{u}) \mathbf{B}_0$$

Ignore dissipation

$$\partial_t \delta \rho + \rho_0 (\nabla \cdot \delta \mathbf{u}) = 0$$

This is a linear problem; that can be solved by Fourier transform and matrices.

$$\delta \mathbf{u} = \tilde{\mathbf{u}} \exp(-i\omega t + i\vec{k} \cdot \vec{x}), \quad \delta \mathbf{b} = \tilde{\mathbf{b}} \exp(-i\omega t + i\vec{k} \cdot \vec{x})$$

$$\delta \rho = \tilde{\rho} \exp(-i\omega t + i\vec{k} \cdot \vec{x})$$

Typically takes the form:

Also

$$M(\psi) = 0$$

A solution exists only when $\det M = 0$

$$\begin{pmatrix} -i\omega & * & * & * & * \\ & * & * & * & * \\ * & & & & \\ * & & & & \\ * & & & & -i\omega \end{pmatrix} \begin{pmatrix} \tilde{u}_x \\ \tilde{u}_y \\ \tilde{u}_z \\ \tilde{\rho} \\ \tilde{b}_x \\ \tilde{b}_y \\ \tilde{b}_z \end{pmatrix} = 0$$

