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# Some solutions of MHD equations

~~4. Comment~~

1. Eqn of isothermal MHD

$$\rho\left[\partial_t u + (u\cdot\nabla)u\right] = \nu\nabla^2 u - \nabla p + \frac{J\times B}{\mu_0}$$

$$\partial_t\rho + \nabla\cdot(\rho u) = 0$$

$$\partial_t B + (u\cdot\nabla)B = (B\cdot\nabla)u - (\nabla\cdot u)B$$

$$p = c_s^2\rho$$

· comments

1. non-linearity
2. Existence of solutions.
3. Hydrynamic limit, and complexity of solutions, connetion with turbulence.

$$\left.\begin{array}{l}\partial_t u + (u\cdot\nabla)u = \nu\nabla^2 u - \nabla p \\ + \ \nabla\cdot u = 0\end{array}\right\}$$

4. Turbulence is ubiquitous.

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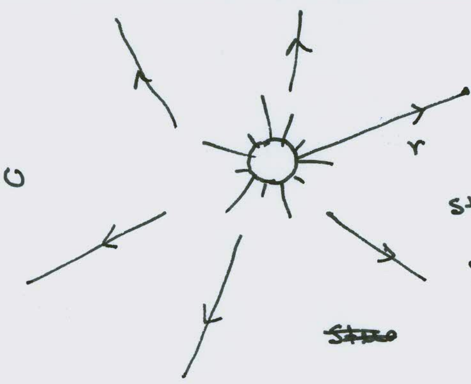
2. How to solve eqns of MHD.

Step 1. Look for symmetry, $\vec{B}=0$,

$u_r = u, \quad u_\theta, u_\phi = 0, \quad \partial_\theta = 0, \quad \partial_\phi = 0$




steady sol$^n$

$\partial_t u = 0$

$$\Rightarrow \quad \rho u\frac{du}{dr} = -\frac{dp}{dr} - \frac{GM}{r^2}\rho$$

and $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho u\right) = 0$$

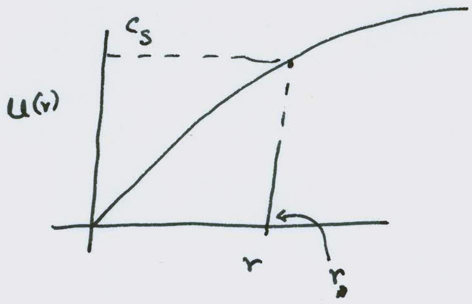
with $p = c_s^2\rho \longrightarrow$ isothermal

Integrate to obtain: $\frac{1}{2}u^2 + c_s^2\ln\rho - \frac{GM}{r} = \text{constant} \equiv \mathcal{E} \leftarrow$ energy

$4\pi r^2\rho u = \text{constant} = \dot{M} \leftarrow$ mass flux

Problem solved! To obtain $u$, eliminate $\rho$, to obtain:

$$\frac{1}{2}u^2 - c_s^2\ln u - 2c_s^2\ln r - \frac{GM}{r} = \mathcal{E}', \qquad \mathcal{E}' = \mathcal{E} - c_s^2\ln\left(\frac{\dot{M}}{4\pi}\right)$$




$$c_s = \left(\frac{GM}{2r_*}\right)^{1/2} \qquad r_* \sim 5R_\odot$$

Parker Wind

3. How to solve eqns of MHD

<u>step 2</u> Stability of the solution.

magnetic pressure

$$\rho\left(\partial_t u + u\cdot\nabla u\right) = \nu\nabla^2 u - \nabla p + \frac{J\times B}{\mu_0} \begin{cases} \nearrow -\frac{1}{2}\nabla B^2 \\ \searrow B\cdot\nabla B \end{cases}$$

$$\partial_t \rho + \nabla\cdot(\rho u) = 0$$

$$\partial_t B + (u\cdot\nabla) B = (B\cdot\nabla) u - (\nabla\cdot u) B + \eta\nabla^2 B$$

$$u \to u_0 + \delta u \;; \quad B \to B_0 + \delta b, \quad \rho \to \rho_0 + \delta\rho$$

Substitute and expand :

$$(u\cdot\nabla) u = (u_0 + \delta u)\cdot\nabla(u_0 + \delta u)$$

$$= (u_0\cdot\nabla) u_0 + (\delta u\cdot\nabla) u_0 + (u_0\cdot\nabla)\delta u + (\delta u\cdot\nabla)\delta u$$

Ignore second order terms (infinitismal pertur bation)

to obtain :

$$\left.\begin{aligned} \rho_0\,\partial_t \delta u &= -c_s^2\nabla\delta\rho - \nabla\left(\frac{B_0\cdot\delta b}{\mu_0}\right) + \frac{1}{\mu_0}(B_0\cdot\nabla)\delta b \\ \partial_t \delta b &= (B_0\cdot\nabla)\delta u - (\nabla\cdot\delta u) B_0 \end{aligned}\right\} \text{Ignore dissipation}$$

$$\partial_t \delta\rho + \rho_0(\nabla\cdot\delta u) = 0$$

This is a linear problem; that can be solved by Fourier transform and matrices.

$$\delta u = \tilde{u}\exp(-i\omega t + i\vec{k}\cdot\vec{x}), \quad \delta b = \tilde{b}\exp(-i\omega t + i\vec{k}\cdot\vec{x})$$

$$\delta\rho = \tilde{\rho}\exp(-i\omega t + i\vec{k}\cdot\vec{x})$$

Typically takes the form :

$$\begin{pmatrix} -i\omega & \times & & \times & \times & \times & \\ & \ddots & \times & & \times & & \\ \times & & \ddots & & & & \\ & & & \ddots & \times & & \\ \times & & & & \ddots & & \\ & & \times & & & \ddots & \\ \times & & & & & & -i\omega \end{pmatrix} \begin{pmatrix} \tilde{u}_x \\ \tilde{u}_y \\ \tilde{u}_z \\ \tilde{\rho} \\ \tilde{b}_x \\ \tilde{b}_y \\ \tilde{b}_z \end{pmatrix} = 0$$

~~Also~~

$$M|\psi\rangle = 0$$

A solution exists only when $\det M = 0$

$\det M = 0 \Rightarrow$ (a 7th order polynomial in $\omega$) $= 0$

consider a simple example : $B_0 = \hat{z} B_0$, $u_0 = 0$, $\rho_0 =$ constant

and assume we live in 1-d space, i.e. $\vec{k} = (0, 0, k_z)$

~~Then~~ or $\partial_z =$ nonzero, $\partial_x = 0$, $\partial_y = 0$.

The problem simplifies to a simpler eqns:

$\omega = 0$, $\omega^2 - c_s^2 k^2 = 0$ $\longrightarrow$ sound waves

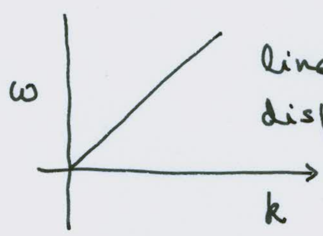
$\omega^4 - 2 v_A^2 k^2 \omega^2 + v_A^4 k^4 = 0$ (7 roots in total)

$\downarrow$

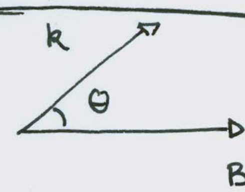
$(\omega^2 - v_A^2 k^2)^2 = 0$ Alfven wave.

Transver wave. magnetic field behaves like a string

with $v_A = \dfrac{B_0}{\sqrt{\rho_0 \mu_0}}$.



linear dispersion relation.



In a more complex 3d case one obtains

$\omega = 0$, $\omega^4 - \omega^2 k^2 (v_A^2 + c_s^2) + k^4 v_A^2 c_s^2 \cos\theta = 0$, $\omega^2 + k^2 v_A^2 = 0$

$\downarrow$ fast and slow magnetosonic wave (two roots each $\}$, 4 in total)

$\uparrow$ Alfven wave (two roots)

for $\theta = 0$

$(\omega^2 - c_s^2 k^2)(\omega^2 - k^2 v_A^2) = 0$

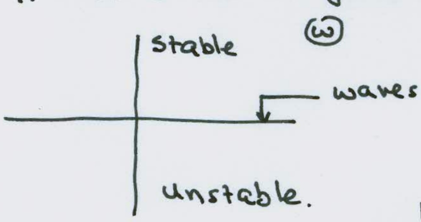
sound Alfven

comments

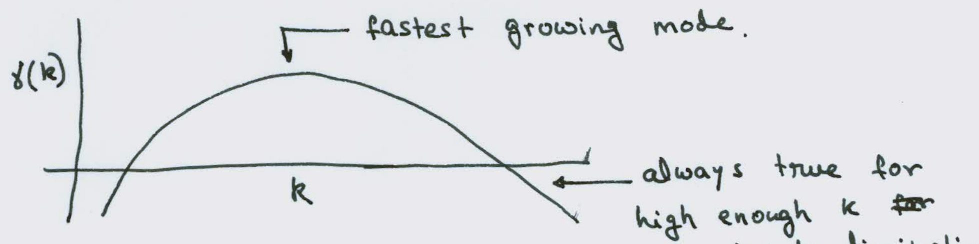
1. $\omega$ could in general be: real : waves

+ve Imaginary : $i\gamma(k) \rightarrow$ decaying sol$^n$

-ve imaginary : $-i\gamma(k) \rightarrow$ growing sol$^n$



In general for unstable case:



Mechanism of pattern formation in complex systems.

An important example is the magnetorotational ~~ist~~ instability (MRI)

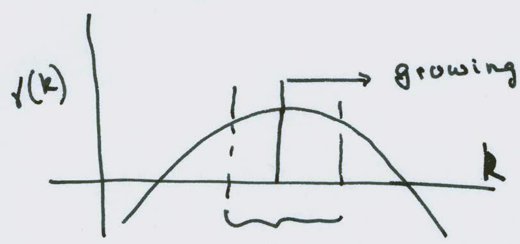
2. For complex geometries, even the linear equations cannot be solved analytically. One ~~may~~ must construct, numerically, the matrix M and then ~~co~~ numerically diagonalize it. This is a field of active research. ~~One~~ ~~ca~~ We can make significant progress if we try to look for only the fastest growing mode.

3. The linear theory is not very useful. Linearly stable flows can be actually unstable. A typical example is the pipe flow, which is linearly stable but of course turbulent in practice.

4. Saturation cannot be described by linear theory. There may be ~~scond~~ secondary instabilities. It is possible to study beyond linear stability by weakly nonlinear perturbative approach.

$$\partial_t \psi = \gamma \psi \quad \leftarrow \text{linear growth.}$$

Note the need for seed fields




→ growing solution.

→ expand around the growing solution and keep terms upto next order

$$\partial_t \psi = \gamma \psi - \mu \psi^2$$

↑ saturation.

~~The saturation~~ can ~~be determined~~

5. Pay attention to dissipation. It cannot be arbitrarily set to zero. $\nu = 0$, and $\nu \to 0$ limit can be very different. $\eta \to 0$ is the limit of reconnection.

4. How to solve eqn of MHD

<u>step 3</u> Give up and use a computer.

There exists a kaleidoscope of codes; each working in a limited domain, all limited by our computing power. For example, the simulation of the sun corresponds to a sun, not filled with plasma, but with honey. see the talk by Prateek sharma.

To summarize: our knowledge of the solutions of MHD eqn is very limited. The main research question is how can we describe the natural phenomenon as solutions of the MHD equations.

## References


- Astrophysical fluids and plasmas – by Arnab Rai Chaudhuri. Possibly the best introductory book in this subject.
- But be sure to read Parker's original paper on the solar wind problem.
- About MRI, read the paper by Balbus and Hawley in Rev. Mod. Phys.
- As an example of application of weakly nonlinear approach to study saturation of a ~~by~~ MHD instability see arXiv: 1011.1251