Chapter 3

Measurement of Polarization

Having discussed the details of design and construction of the instrument in the previous chapter, here we consider what the best way is of using it for polarimetric measurements. In order to arrive at an optimum method, firstly, we examine the nature of various uncertainties associated with the determination of polarization. On the basis of these findings we arrive at a strategy for making observations with the instrument and analyzing the data. The practical details of this strategy, in terms of the sequence of steps to be followed for observing with IMPOL, have been listed out in App. D and App. E.

3.1 Sources of measurement errors

In addition to the fundamental limit due to photon statistics, the measurement of polarization is affected by several other factors, beginning from the atmosphere down to the detector and the data-analysis procedure. Here, we make quantitative estimates of how these errors affect the measurement of polarization. We see that photon noise associated with the measurement of intensities is the major contributor of noise in typical polarimetric observations, unless accuracies better than 0.1% are sought. In section 3.1.5 we estimate the accuracies which the instrument is expected to achieve for two typical types of observations, assuming that the noise is solely due to photon statistics.

3.1.1 Atmospheric effects

Since polarization measurement involves taking multiple exposures, variations in atmospheric transparency and scintillations can affect the measurement process. In IMPOL, two orthogonal polarization components are measured simultaneously and the Stokes parameter is obtained from the ratio of the fluxes in these components (Eq. 2.4). Since the atmosphere is not birefringent, this eliminates the effects due to atmospheric scintillation, or for that matter any effect which changes both the polarization components by the same factor–like variations in the effective exposure times of observations, presence of thin clouds etc.

3.1.2 Distortions in polarization within the instrument

There are a number of issues associated with the instrument directly, which affect the measurement of polarization. As the measurements depend on detecting changes in the polarization vector caused by rotating the half-wave plate, any small instrumental polarization occurring after the half-wave plate can be easily eliminated. All the optical elements are anti-reflection coated to minimize the polarization effects due to reflection at their surfaces, and care is taken to minimize the stray light, reflected from the walls etc., reaching the detector. In particular, all the aperture stops in the beam path are chosen to be non-metallic, including the grid in the focal plane, so as to avoid polarization of light scattered from these (Pospergelis, 1965). The mounts for field lens, the half-wave plate and the Wollaston prism have been designed to minimize stress-birefringence due to differential thermal expansion. The filter is inserted after the analyzer so that any birefringence in this component does not affect the measurement.

The retardance introduced by the half-wave plate could deviate from 180° either because of the finite angle of incidence or because of the chromatic effects. If the beam is incident at a small angle i, the maximum change in retardance is given by (derived from expression in Serkowski 1975)

$$\sigma_\tau \simeq \pi \times \frac{i^2}{4n_\mathrm{o}}\left(\frac{1}{n_\mathrm{o}} - \frac{1}{n_\mathrm{e}}\right) , \tag{3.1}$$

where n_o and n_e are the refractive indices of the material for ordinary and extraordinary rays. For the aperture used, the maximum angle of incidence is about 5° so that $\sigma_\tau \lesssim 0.01$ rad. The chromatic effects give $\sigma_\tau < 0.18$ rad for the wideband and $\sigma_\tau < 0.14$ rad for the V-band. It can be shown (Serkowski 1974) that the depolarization σ_p due to an uncertainty of σ_τ rad in the retardance is to the lowest order given by $\sigma_p \simeq p\sigma_\tau^2$, p being the fractional polarization. Therefore even for the wideband $\sigma_p \lesssim 0.03p$. Further, circular polarization in the incident light is converted to a linear polarization of magnitude $V\sigma_\tau$, where V is the circularity parameter. For wideband observations, conversion to linear polarization would be about $0.18V$, but in typical observations this does not pose a serious problem because the circular polarization in the incident light is usually less than the linear polarization.

Another chromatic effect is the change in position angle of the half-wave plate fast axis with wavelength. For wideband observations, this renders the measurement of both the fractional polarization p and the position angle θ, erroneous. A solution to this problem is to observe standard stars using narrow band filters and produce a smoothed curve which gives the required position angle correction for each wavelength. The achromatic half-wave plate used in IMPOL does not produce any appreciable dispersion in its fast-axis position angle over the wavelength range of interest.

The uncertainty involved in the positioning of the half-wave plate leads to an error in the measurement of polarization. From equation 2.4, we can see that a positioning uncertainty of σ_α produces a maximum error in the measurement of linear polarization given by $\sigma_p \simeq p\sigma_\alpha$. Thus for an error of 0.1°, of the positioning system, this results in $\sigma_p \simeq 0.002p$.

3.1.3 Flat field errors

We have seen in Sect. 2.2 that observations for at least two positions of the half-wave plate are required to determine the three unknown parameters namely, the total, intensity (I), fraction of light in linearly polarized condition (p), and position angle of the plane of polarization (θ). However, the situation is not so simple in reality because of two reasons – (i) the responsivity of the system to the two orthogonal polarization components may not be the same, and (ii) the responsivity of the CCD is a function of the position on its surface. Due to these effects the signals which are actually measured in the two images (I'_e and I'_o) are given by

$$\begin{aligned} I'_\mathrm{e}(\alpha) &= I_\mathrm{e}(\alpha)F_\mathrm{e} \quad \text{and} \\ I'_\mathrm{o}(\alpha) &= I_\mathrm{o}(\alpha)F_\mathrm{o}\,, \end{aligned} \tag{3.2}$$

where $I_\mathrm{e}(\alpha)$ & $I_\mathrm{o}(\alpha)$ are the flux values in the images and F_e & F_o represent the effects mentioned above. In order to ensure that F_e and F_o do not change during observations of an object, the positions of the two images are kept fixed on the CCD. Further, as the analyzer is fixed with respect to the CCD, the orientation of the ordinary and extraordinary polarizations also remain fixed with respect to the CCD, independent of the polarization vector of the incident light. Therefore the ratio of the factors F_o and F_e can be estimated as

$$\frac{F_\mathrm{o}}{F_\mathrm{e}} = [\frac{I'_\mathrm{o}(0°)}{I'_\mathrm{e}(45°)} \times \frac{I'_\mathrm{o}(45°)}{I'_\mathrm{e}(0°)} \times \frac{I'_\mathrm{o}(22.5°)}{I'_\mathrm{e}(67.5°)} \times \frac{I'_\mathrm{o}(67.5°)}{I'_\mathrm{e}(22.5°)}]^{\frac{1}{4}}\,, \tag{3.3}$$

by making use of the fact that a rotation of the half-wave plate by 45° simply leads to an interchange of the signals in the ordinary and extraordinary images. Now the actual ratio of the fluxes in the two images may be recovered as

$$\frac{I_\mathrm{e}(\alpha)}{I_\mathrm{o}(\alpha)} = \frac{F_\mathrm{o}}{F_\mathrm{e}} \times \frac{I'_\mathrm{e}(\alpha)}{I'_\mathrm{o}(\alpha)}\,. \tag{3.4}$$

Thus we see that by combining the intensities measured at four orientations of the HWP, namely 0°, 22.5°, 45°and 67.5°, as in Eq. 3.3, we can estimate the channel to channel sensitivity variation and remove its effect from the determination of $R(\alpha)$s.

As was emphasized earlier in this section, for the above technique to work, it is necessary to decouple the effects of intrapixel sensitivity variations from channel to channel sensitivity variation. The effects of intrapixel as well as pixel to pixel sensitivity variation, can be eliminated, if the images are stable to within a small fraction of a pixel during exposures at different positions of the half-wave plate. Though, a good telescope tracking system can provide sub-pixel positioning accuracy, effects of any residual image motion can be reduced to acceptable levels by dithering the telescope pointing to get an FWHM of the point-spread function greater than 3 pixels (see Sec. 5.1.1). The dithering has to be done at time scales much shorter than the exposure time.

3.1.4 Photon noise

From the above discussion we find that all the systematic sources of errors described do not contribute more than $0.05p$ to the error in the measurement of p. In this section, we make theoretical estimates of the effect of photon noise in the measurement of polarization. The number of photoelectrons collected in the ordinary and the extraordinary images (N_o and N_e) at any position of the half-wave plate gives the ratio

$$R(\alpha) = \frac{N_\mathrm{o} - N_\mathrm{e}}{N_\mathrm{o} + N_\mathrm{e}} \; . \tag{3.5}$$

The error in determining $R(\alpha)$, due to errors associated with the determination of photoelectron counts, is given by

$$(\Delta R)^2 = (\frac{\partial R}{\partial N_\mathrm{o}})^2(\Delta N_\mathrm{o})^2 \quad + \quad (\frac{\partial R}{\partial N_\mathrm{e}})^2(\Delta N_\mathrm{e})^2 \; . \tag{3.6}$$

If it is assumed that the determination of N_o and N_e are affected by photon noise only, we get,

$$(\Delta N_\mathrm{o})^2 = N_\mathrm{o} + N_\mathrm{B} \qquad \text{and} \qquad (\Delta N_\mathrm{e})^2 = N_\mathrm{e} + N_\mathrm{B} \; , \tag{3.7}$$

where N_B is the background photoelectron counts in the region around the ordinary and the extraordinary images (see Sec. 3.2). Substituting Eq. 3.5 and Eq. 3.7 into Eq. 3.6, we get

$$\begin{aligned} (\Delta R)^2 &= \frac{4{N_\mathrm{e}}^2}{(N_\mathrm{o}+N_\mathrm{e})^4}(N_\mathrm{o}+N_\mathrm{B}) + \frac{4{N_\mathrm{o}}^2}{(N_\mathrm{o}+N_\mathrm{e})^4}(N_\mathrm{e}+N_\mathrm{B}) \\ &= \frac{4N_\mathrm{o}N_\mathrm{e}}{(N_\mathrm{o}+N_\mathrm{e})^3} + \frac{4N_\mathrm{B}}{(N_\mathrm{o}+N_\mathrm{e})^4}({N_\mathrm{o}}^2+{N_\mathrm{e}}^2) \end{aligned} \tag{3.8}$$

Now, we have to understand how this error affects the determination of p and θ. The equations which govern this are given in Eq. A.3, from which we get

$$p = \sqrt{q^2+u^2} \qquad \text{and} \qquad \theta = \tan^{-1}\frac{u}{q} \; , \tag{3.9}$$

where we already know, $q = R(0°)$ and $u = R(22.5°)$. The error which is associated with p is given by

$$\begin{aligned} (\Delta p)^2 &= (\frac{\partial p}{\partial q})^2(\Delta q)^2 + (\frac{\partial p}{\partial u})^2(\Delta u)^2 \\ &= (\frac{q}{p})^2(\Delta q)^2 + (\frac{u}{p})^2(\Delta u)^2 \; , \end{aligned} \tag{3.10}$$

where the variances $(\Delta q)^2$ and $(\Delta u)^2$ are determined by equations of the form given in Eq. 3.8. Similarly,

$$\begin{aligned}(\Delta\theta)^2 &= (\frac{\partial\theta}{\partial q})^2(\Delta q)^2 + (\frac{\partial\theta}{\partial u})^2(\Delta u)^2 \\ &= \frac{1}{4p^2}[(\frac{u}{p})^2(\Delta q)^2 + (\frac{q}{p})^2(\Delta u)^2] \qquad (3.11)\end{aligned}$$

for θ expressed in radians.

For small values of p, N_o and N_e are approximately equal and denoting $N = N_\mathrm{o} + N_\mathrm{e}$, and $N_\mathrm{b} = 2N_\mathrm{B}$, Eq. 3.8 reduces to the simple form

$$(\Delta R)^2 = \frac{N + N_\mathrm{b}}{N^2} \,. \qquad (3.12)$$

Since this expression depends only on the total number of photoelectrons in the image and the background, it is independent of the position angle of the half-wave plate. Therefore, $(\Delta q)^2 \sim (\Delta u)^2$ and Eq. 3.10 and Eq. 3.11 reduce to

$$(\Delta p)^2 = (\Delta R)^2 \qquad \text{and} \qquad (\Delta\theta)^2 = \frac{1}{4}(\frac{\Delta p}{p})^2 \,, \qquad (3.13)$$

and hence for small polarizations, the error in the measurement of p and θ are given by

$$\sigma_p = \frac{\sqrt{N + N_\mathrm{b}}}{N} \qquad \text{and} \qquad \sigma_\theta = 0.5 \times \frac{\sigma_p}{p} \text{ rad.} \qquad (3.14)$$

The estimates in Eq. 3.14, are exact in the case of unpolarized light, while for polarization values $\lesssim 10\%$, the error in $\frac{\sigma_p}{p}$ when this approximation is used is less than 4%.

It is worth noting here that p, as defined in Eq. 3.9 above, is a positive definite quantity and follows the Rice probability distribution given by

$$F(p, p_0) = \frac{p}{\sigma_p} e^{\frac{p^2+p_0^2}{2\sigma_p^2}} I_0(\frac{pp_0}{\sigma_p^2}) \,. \qquad (3.15)$$

Here, p_0 is the true value of fractional polarization being estimated by p and I_0 is the modified Bessel function of order zero. Since p is a biased estimator, several schemes have been suggested (Simmons & Stewart 1984 and references therein) for debiasing, but none of them is fully satisfactory. For values of $\frac{p}{\sigma_p}$ larger than about 4, almost all the debiasing schemes agree and reduce approximately to the relation

$$\hat{p}_0 = p\sqrt{1 - \frac{\sigma_p^2}{p^2}} \,, \qquad (3.16)$$

where $\hat{p}_0$ is the "debiased" estimate of p_0. Thus it is desirable to work with the normalized Stoke's parameters as far as possible and use the quantities p and θ only to present the final results. However, in general, the normalized Stoke's parameters themselves might not be normally distributed and might have bias (Clarke et. al. 1983). Besides, for photon-noise dominated measurements, the positive kurtosis of the distribution will lead to erroneous estimates of the confidence levels unless more than a few thousand photoelectrons are collected. Thus, in order to arrive at an optimum procedure, it is essential to carefully study the nature of the dominant sources of noise in the measurement and their effects on the analysis.

3.1.5 Performance of Polarimeter

In this section, we calculate the accuracy of polarization measurement due to photon noise for two typical types of observations. It is assumed that the observations are made in the V-band ($V_0 = 10^6$ photons cm^{-2} s^{-1}) using a telescope of effective area $\sim 10^4$ cm^2, the brightness of the sky is 20 mag. per sq. arcsec, and that the software aperture used for photometry is about 30 sq. arcsec. The combined effect of the atmosphere, the telescope, the optics and quantum efficiency of the detector etc., is taken to result in an effective transmission of $\eta \sim 3.2\%$.

A classical example of polarimetric observations involves study of dark molecular clouds with the light of stars shining behind their periphery (eg. Elvius 1970; Joshi et al. 1985; Kane et al. 1995). Typically the polarization of these sources is in the range 1–5 %. Assuming stars of apparent visual magnitude $m_{\mathrm{v}} = 15$, we see that the total number of photoelectrons collected, due to the star (N) and the background (N_B), within the software aperture, for an exposure of 15 minutes are 2.9×10^5 and 8.6×10^4 respectively. Therefore, the corresponding error in polarization measurement is given by

$$\sigma_p = 100 \times \frac{\sqrt{2.9 \times 10^5 + 8.6 \times 10^4}}{2.9 \times 10^5} \simeq 0.21\% \; . \tag{3.17}$$

If the observations are made with a filter which covers both the V and R bands ($\Delta\lambda \sim 200$ nm), there will be an improvement in the signal-to-noise ratio by a factor of $\sqrt{2}$, giving $\sigma_p \sim 0.15\%$.

Study of extended objects like reflection nebulae, is another field in which polarimetric observations are useful. Polarization in the light scattered from these clouds can range from a few percent to as much 30%. Assuming that the observation involves a reflection nebula of the same surface brightness as the background sky, the integrated photoelectron count from each, for an exposure of 15 minutes, will be 8.6×10^4 and hence

$$\sigma_p = 100 \times \frac{\sqrt{8.6 \times 10^4 + 8.6 \times 10^4}}{8.6 \times 10^4} \simeq 0.48\% \; . \tag{3.18}$$

for a solid angle defined by the software aperture. The wideband observations, in this case will give an error of about 0.34%.

3.2 Data-analysis

In Sec. 3.1.3, it was seen that measurements at four HWP orientations are required to eliminate the flat-field errors, before determining the normalized Stokes parameters – $R(\alpha)$s. Each of the two pairs of normalized Stokes parameters $(R(0°), R(22.5°))$ and $(R(45°), R(67.5°))$, can be combined as shown in Eq. C.11, giving two estimates each for p and θ. Alternatively, the maximum likelihood estimator for polarization can be determined by fitting a cosine curve of the form given in Eq. C.10, to the four normalized Stokes parameters simultaneously, using χ^2-minimization techniques. However, in order to obtain correct estimates of the errors in determining p and θ and to estimate the goodness of fit, one needs an estimate of the uncertainty in the measurement of $R(\alpha)$s. In view of the importance of making the correct error estimates, we defer the discussion of this issue till the next subsection, where it will be considered in detail.

The tools for analysing the data has been developed within the *Image Reduction and Analysis Facility* (IRAF[1]). The polarimetry package accepts a set of frames arranged as quadruplets of the four HWP orientations. Photometry is carried out on each of these frames within apertures of specified radius and central pixels, to determine the counts due to both the object and the background. Then ordinary and extraordinary image pairs are identified in each of the frames and the normalized Stokes parameters determined. A cosine curve is fitted to all the Stokes parameter values for a given object, to determine the values of p and θ.

3.2.1 A few important considerations during data-analysis

To a large extent, the accuracy of polarimetry depends on the details of the techniques employed for the analysis of the raw data and extraction of the polarimetric information. Some of these issues are discussed below.

Determination of background (N_{B})

Background, in this context, is an estimate of the number of photoelectrons collected within the software aperture used for photometry, from sources other than the object of interest. The usual practice is to identify a region in the frame, around or near the image of the object of interest and use some estimator of the central tendency of photoelectron counts within this region, as the value of the background per pixel. While selecting such a region, it should be ensured that all sources which contaminate the photoelectron counts within the aperture are taken care off.

Various estimators of central tendency like the mode, median, mean etc. have

[1]IRAF is distributed by National Optical Astronomy Observatories, which are operated by the Association of Universities for Research in Astronomy, Inc, under contract with the National Science Foundation.

been used for determining the background. If the majority of the pixels in the region from where the background is estimated, is not contaminated by other nearby sources, the mode is the most robust estimator. But, in case there is a sharp variation of the background in the region, then median or mean would be a better choice. In our experience, median seems to be the better estimator, particularly, for stellar fields where stars are present close to the edges of the field.

We see from Eq. 3.14 that it is essential to keep the background as low as possible to achieve good accuracy of polarimetry, particularly when observing faint targets. Quite often, while observing stellar fields, in order to reduce observing time, the grid at the front of the optics (see Fig. 2.11) is removed so that the entire field is available for polarimetry. One of the disadvantages of this technique is that the ordinary and the extraordinary images of the sky overlap and hence the effective background becomes double.

Choice of photometry aperture

Proper choice of apertures, for estimating the photoelectron counts from the object as well as the sky, is extremely necessary to ensure that the noise from background is minimized, while at the same time maximum use of the signal from the object, is made. If we draw a graph showing the total number of object-counts within an aperture against the radius of the aperture, it will grow initially, but starts flattening after some radius. Signal counts do not gain much by increasing the aperture beyond this radius while at the same time there will be added contribution to the noise from the background. The choice of aperture radius depends also on the consideration that the effect of any changes in the point-spread function due to image motion etc. should be negligible. Thus for bright stars it would be advantageous to detect a large aperture radius, whereas for fainter stars a small aperture radius should be selected.

A useful estimate of the aperture radius is the FWHM of the point-spread function. For typical images of bright stars obtained with IMPOL, it will be about 3 pixels and contains about 95% of the flux due to the star, whereas for faint stars an aperture of about 2.5 pixel radius will contain about 90% of the light. The background is determined from an annular aperture around each star, of inner radius 5 pixels and outer radius 11 pixels.

Combining the raw data

It is useful to break up the exposure needed at each HWP position into a few ($\lesssim 10$) shorter exposures. Thus, even if one of the exposures gets corrupted, for example by a cosmic ray event, the remaining data could be used for extracting polarimetric information. Besides, the standard deviation of the multiple values of the ratio $R(\alpha)$ obtained at each HWP position, provides an estimate of the actual errors in the determination these ratios. This estimate is used by the χ^2-minimization technique to determine the errors in p and θ obtained through fitting. It is also possible to get an estimate of the errors expected on the basis of photon statistics alone, as in the examples given in Sec. 3.1.5. A useful consistency check is to ensure that estimates of uncertainties reported by the fitting process match within statistical errors with those calculated on the basis of photon-statistics.

All estimates of the significance and confidance levels of polarization measurement assume that the uncertainties involved are such that the basic measurements are gaussian random distributed. But this is not always the case, particularly, when the photoelectron counts are few. To ensure that the normalized Stoke's parameters follow a normal distribution as closely as possible, the signals measured at each position of the half-wave plate, should be averaged together before combining them to give the normalized Stoke's parameters. However, a disadvantage of this method is that it gives only one degree of freedom for estimating the two parameters p and θ from the four values of $R(\alpha)$. A better approach is to estimate $R(\alpha)$ from each of the exposures and then to estimate p and θ by fitting a cosine curve to all the values of $R(\alpha)$ simultaneously.

Polarization position angle θ

Within the instrument all angles (position angles of polarization components of light, orientation of the HWP fast-axis etc.) are measured with respect to a particular reference direction. This is the direction in which the ordinary and extraordinary rays are separated as they emerge from the Wollaston prism. The polarimetric position angles obtained from measurements using the instrument are all with reference to this direction. However, the universally accepted convention is to report the polarization position angle as referred to the celestial north-south axis passing through the point in the sky where the measurement is made – $0°$ is towards north and $90°$ towards east; the position angle is quoted in the range $0° < \theta < 180°$.

To conform with this convention, it is convenient if the instrument is mounted on the telescope in such a way that the reference direction within the instrument, mentioned above, is aligned with the declination axis of the telescope. Besides, standard polarized stars, having known position angles should be observed with the instrument and correction for any residual misalignment estimated. This estimate is to be used to correct all polarization position angles measured by the instrument. Aligning the instrument's reference axes with those of the telescope is necessary for proper working of the acquisition and guidance system of the instrument (see Sec. 4.3.1).

A relevant point to be noted in this context is that due to an oversight during the design of the instrument, the position angle of the HWP is measured with a convention slightly different from what has been described above. With the present arrangement, what is referred to as angle α for the HWP is actually $\alpha' = -\alpha$ according to the above convention. The only difference which this makes is that the basic equation given in Eq. 2.4 should be modified as

$$\frac{I_{\rm o}/I_{\rm e} - 1}{I_{\rm o}/I_{\rm e} + 1} = R(\alpha) = p\cos(2\theta - 4\alpha')\ . \qquad (3.19)$$

Table 3.1: Convention for calculating polarization angles from instrumental angles

$\theta_{\rm calc} = \frac{1}{2}\tan^{-1}[\frac{\lvert R(22.5^\circ)\rvert}{\lvert R(0^\circ)\rvert}]$		
$R(0^\circ)$	$R(22.5^\circ)$	θ
$+$	$-$	$\theta_{\rm calc}$
$-$	$-$	$90^\circ - \theta_{\rm calc}$
$-$	$+$	$90^\circ + \theta_{\rm calc}$
$+$	$+$	$180^\circ - \theta_{\rm calc}$

$\theta_{\rm calc} = \frac{1}{2}\tan^{-1}[\frac{\lvert R(67.5^\circ)\rvert}{\lvert R(45^\circ)\rvert}]$		
$R(45^\circ)$	$R(67.5^\circ)$	θ
$-$	$+$	$\theta_{\rm calc}$
$+$	$+$	$90^\circ - \theta_{\rm calc}$
$+$	$-$	$90^\circ + \theta_{\rm calc}$
$-$	$-$	$180^\circ - \theta_{\rm calc}$

The equation for determining θ from $R(\alpha)$s is given in Eq. C.11. Table 3.1 shows how to convert the instrumental angles (when the instrument is mounted on the telescope as shown in Fig. D.1) to the universal convention described above.